Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 6
Solutions

1. Suppose

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

is a short exact sequence of chain complexes and chain mappings.

Consider the sequence

$$\dots \longrightarrow H_{n+1}(\overline{C}) \xrightarrow{\partial} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(\overline{C}) \xrightarrow{\partial} H_{n-1}(C') \longrightarrow \dots,$$

where ∂ is a boundary homology operator, defined as usual.

a) Prove that

$$\operatorname{Ker} \partial \subset \operatorname{Im} g_*.$$

b) Prove the exactness of the sequence at $H_n(C')$.

Solution: a) Suppose $[x] \in H_n(\overline{C})$ is an element of $\operatorname{Ker} \partial$. Hence $x \in Z_n(\overline{C})$ and $\partial[x] = 0 \in H_{n-1}(C')$. By the definition of boundary operator $\partial[x] = [z]$, where $f_{n-1}(z) = \partial_n(y)$ for some $y \in C_n$ with $g_n(y) = x$. Now $[z] = 0$, hence $z \in B_n(C')$ i.e. there exists $u \in C'_n$ such that $\partial'_n(u) = z$. By commutativity we have

$$\partial_n(f_n(u)) = f_{n-1}(\partial'_n(u)) = f_{n-1}(z) = \partial_n(y).$$

It follows that $\partial_n(y - f_n(u)) = 0$, hence $y - f_n(u) \in \operatorname{Ker} \partial_n = Z_n(C)$ and the homology class $[y - f_n(u)] \in H_n(C)$ exists. Moreover

$$g_*([y - f_n(u)]) = [g_n(y) - g_n(f_n(u))] = [g_n(y)] = [x],$$

since $g \circ f = 0$ by exactness. In particular $[x] \in \operatorname{Im} g_*$.

b) Suppose $[x] \in H_n(\overline{C})$, where $x \in Z_n(\overline{C})$. Then there exists $y \in C_n$ and $z \in C'_{n-1}$ such that $g_n(y) = x$ and $f_{n-1}(z) = \partial_n(y)$. Then by definition $\partial([x]) = [z]$, hence

$$f_*(\partial([x])) = f_*(z) = [f_{n-1}(z)] = [\partial_n(y)] = 0 \in H_n(C).$$

In other words $f_* \circ \partial = 0$.

On the other hand suppose $[z] \in H_n(C'), z \in Z_n(C')$ is such that $f_*([z]) = 0 \in H_{n-1}(C)$. This means that $f_{n-1}(z)$ is a boundary element in C_{n-1}, i.e. there exists $y \in C_n$ such that $\partial_n(y) = f_{n-1}(z)$. Let $x = g_n(y) \in \overline{C}$. Then by commutativity and exactness

$$\overline{\partial}(x) = g_{n-1}(\partial(y)) = g_{n-1}(f_{n-1}(z)) = 0.$$

Hence $x \in Z_n(\overline{C})$ and by construction $\partial[x] = [z]$. We have shown that

$$\operatorname{Ker} f_* \subset \operatorname{Im} \partial.$$

2. Suppose (X, A, B) is a topological triple.
a)Prove that

$$0 \longrightarrow C(A,B) \xrightarrow{i_\sharp} C(X,B) \xrightarrow{j_\sharp} C(X,A) \longrightarrow 0,$$

where $i\colon (A,B) \to (X,B)$ and $j\colon (X,B) \to (X,A)$ are obvious inclusions, is a short exact sequence. Deduce the existence of the long exact sequence

$$\ldots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial'} H_n(A,B) \xrightarrow{i_*} H_n(X,B) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \longrightarrow \ldots.$$

b) Show that for the boundary operators of the long exact homology sequences of the pair (X, A) and of the triple (X, A, B) there is a commutative diagram

$$\begin{array}{ccc} & & H_n(A) \\ & \overset{\partial}{\nearrow} & \\ H_{n+1}(X,A) & & \downarrow i_* \\ & \underset{\partial'}{\searrow} & \\ & & H_n(A,B), \end{array}$$

where $i\colon A \to (A,B)$ is an inclusion. (Hint: use naturality of the long exact homology sequence.)

Solution: a)

$$\operatorname{Im} i_\sharp = \{x + C(B) \in C(X)/C(B) \mid x \in C(A)\} = C(A)/C(B) \subset C(X)/C(B),$$

on the other hand

$$\operatorname{Ker} j_\sharp = \{x+C(B) \in C(X)/C(B) \mid x+C(A) = 0 \in C(X)/C(A)\} = C(A)/C(B) \subset C(X)/C(B).$$

Hence sequence is exact at $C(X,B)$.
$j_\sharp$ is clearly a surjection, since $j_\sharp(x + C(B)) = x + C(A)$ for every $x \in C(X)$.
Also $i_\sharp$ is an injection, since if $i_\sharp(x + C(B)) = x + C(B) = 0 \in C(X)/C(B)$, this implies that $x \in C(B)$, so $x + C(B) = 0 \in C(A)/C(B)$.

We have shown that the sequence

$$0 \longrightarrow C(A,B) \xrightarrow{i_\sharp} C(X,B) \xrightarrow{j_\sharp} C(X,A) \longrightarrow 0,$$

is exact.

Theorem 2.2.5 implies now the existence of the long exact sequence

$$\ldots \longrightarrow H_{n+1}(X,A) \xrightarrow{\partial'} H_n(A,B) \xrightarrow{i_*} H_n(X,B) \xrightarrow{j_*} H_n(X,A) \xrightarrow{\partial} H_{n-1}(A,B) \longrightarrow \ldots.$$

b)Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(A) & \xrightarrow{i_\sharp} & C(X) & \xrightarrow{j_\sharp} & C(X,A) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle k_\sharp} & & \downarrow{\scriptstyle l_\sharp} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
0 & \longrightarrow & C(A,B) & \xrightarrow{i_\sharp} & C(X,B) & \xrightarrow{j_\sharp} & C(X,A) & \longrightarrow & 0,
\end{array}$$

where $k\colon A \to (A,B)$ and $l\colon X \to (X,B)$ are inclusions. It is easy to verify that the diagram is commutative. By the naturality of the boundary operator (Lemma 2.2.4) the diagram

$$\begin{array}{ccc}
H_{n+1}(X,A) & \xrightarrow{\partial} & H_n(A) \\
\downarrow{\scriptstyle \mathrm{id}_*} & & \downarrow{\scriptstyle k_*} \\
H_{n+1}(X,A) & \xrightarrow{\partial} & H_n(A,B)
\end{array}$$

is commutative. Since $\mathrm{id}_* = \mathrm{id}$ the claim follows.

3. Prove the second part of the Five-Lemma: Suppose the diagram of groups and homomorphisms

$$\begin{array}{ccccccccc}
G_1 & \xrightarrow{\alpha_1} & G_2 & \xrightarrow{\alpha_2} & G_3 & \xrightarrow{\alpha_3} & G_4 & \xrightarrow{\alpha_4} & G_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
H_1 & \xrightarrow{\beta_1} & H_2 & \xrightarrow{\beta_2} & H_3 & \xrightarrow{\beta_3} & H_4 & \xrightarrow{\beta_4} & H_5
\end{array}$$

is commutative, rows are exact, f_5 is injective and f_2, f_4 are surjective. Then f_3 is surjective.
Solution: Suppose $y \in H_3$. We need to find $x \in G_3$ such that $f_3(x) = y$.

Now f_4 is surjective, so there exists $u \in G_4$ such that

$$f_4(u) = \beta_3(y).$$

By commutativity and exactness we have

$$f_5(\alpha_4(u)) = \beta_4(f_4(u)) = \beta_4(\beta_3(y)) = 0.$$

Since f_5 is injective this implies that $\alpha_4(u) = 0$. By exactness there exists $v \in G_3$ such that $\alpha_3(v) = u$. Now

$$\beta_3(f_3(v)) = f_4(\alpha_3(v)) = f_4(u) = \beta_3(y), \text{ so}$$
$$\beta_3(y - f_3(v)) = 0.$$

By exactness there exists $z \in H_2$ such that $\beta_2(z) = y - f_3(v)$. Since f_2 is a surjection there is $w \in G_2$ such that $f_2(w) = z$. We obtain that

$$f_3(\alpha_2(w) + v) = f_3(\alpha_2(w)) + f_3(v) = \beta_2(f_2(w)) + f_3(v) = y.$$

4. Suppose the chain complex C is a direct sum of complexes $(C_a)_{a\in\mathcal{A}}$. Prove that the inclusion mappings $i_a\colon C_a \to C$ induce a chain isomorphism

$$((i_a)_*)_{a\in\mathcal{A}}\colon \oplus_{a\in\mathcal{A}} H_n(C_a) \to H_n(C)$$

for every $n \in \mathbb{N}$.

Solution: Recall the following fact from the theory of groups. Suppose $(G_a)_{a\in A}$ is a collection of abelian groups and H_a is a subgroup of G_a for every $a\in A$. Let $\pi_a\colon G_a\to G_a/H_a$ be a canonical projection. Then

$$p=\oplus_{a\text{ß}A}p_a\colon\ \oplus_{a\in A}G_a\to\oplus_{a\in A}(G_a/H_a)$$

is clearly surjective and its kernel equals $\oplus_{a\in A}H_a$. Hence p induces natural isomorphism

$$\overline{p}=\oplus_{a\in A}G_a/\oplus_{a\in A}H_a\cong\oplus_{a\in A}(G_a/H_a).$$

The inverse of p is defined as following. Let $i_a\colon G_a\to\oplus_{a\in A}G_a$ be a canonical embedding and $\pi\colon\ \oplus_{a\in A}G_a\to\oplus_{a\in A}G_a/\oplus_{a\in A}H_a$ be a canonical projection. Then the composition $\pi\circ i_a$ induces a mapping $i'_a\colon G_a/H_a\to\oplus_{a\in A}G_a/\oplus_{a\in A}H_a$ and $\oplus i'_a$ is an inverse mapping of *overlinep*.

Now suppose $C=\oplus(C_a)_{a\in\mathcal{A}}$. Then by definition of the boundary operator

$$Z_n(C)=\oplus Z_n(C_\alpha),\ \text{and}$$
$$B_n(C)=\oplus B_n(C_\alpha).$$

Then

$$H_n(C)=Z_n(C)/B_n(C)\cong\oplus_{a\in A}(Z_n(C_\alpha)/B_n(C_\alpha))=\oplus_{a\in A}H_n(C_\alpha).$$

Moreover consideration above show that the isomorphism is exactly $((i_a)_*)_{a\in\mathcal{A}}$.

5. a) Suppose

$$0\longrightarrow C'\xrightarrow{f}C\xrightarrow{g}\overline{C}\longrightarrow 0$$

is a short exact sequence of chain complexes and chain mappings. Prove that if any two of complexes $C',C,\overline{C}$ are acyclic, also the third one is acyclic.

b) Suppose

$$\begin{array}{ccccccccc}
\dots & & 0 & & 0 & & 0 & & \dots\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & A_{n-1} & \longrightarrow & A_n & \longrightarrow & A_{n+1} & \longrightarrow & \dots\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & B_{n-1} & \longrightarrow & B_n & \longrightarrow & B_{n+1} & \longrightarrow & \dots\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & C_{n-1} & \longrightarrow & C_n & \longrightarrow & C_{n+1} & \longrightarrow & \dots\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & & 0 & & 0 & & 0 & & \dots
\end{array}$$

is a commutative diagram of abelian groups and homomorphisms. Assume that all columns are exact and the middle row (of B's) is exact. Prove that the upper row (A's) is exact if and only if lower row (C's) is exact.

Solution: a) Consider the long exact sequence in homology

$$\dots\longrightarrow H_{n+1}(\overline{C})\xrightarrow{\partial}H_n(C')\xrightarrow{f_*}H_n(C)\xrightarrow{g_*}H_n(\overline{C})\xrightarrow{\partial}H_{n-1}(C')\longrightarrow\dots$$

Assumptions imply that this sequence has the following property: every part of three consecutive groups is such that at least two groups in it are trivial.
It is enough to show that any exact sequence with such property contains only trivial groups. Indeed in this group every group A which is not known to be trivial is part of the exact sequence

$$0 \xrightarrow{f} A \xrightarrow{g} 0.$$

Now $A = \operatorname{Ker} g = \operatorname{Im} f = 0$ and we are done.

b) The middle row is exact, so in particular can be thought of as a chain complex B (which is acyclic). It Follows that upper row A can be then considered a subcomplex of B and the lower row C is then a quotient complex B/A. Since B is acyclic, the claim follows from a).

6. Suppose $f\colon X \to Y$ is continuous and X, Y are both path-connected and non-empty spaces. Show that $f_*\colon H_0(X) \to H_0(Y)$ is an isomorphism.

   Solution: Consider the commutative diagram

$$\begin{array}{ccc} H_0(X) & & \\ & \searrow^{\varepsilon_*} & \\ \big\downarrow f_* & & \mathbb{Z} \\ & \nearrow_{\varepsilon_*} & \\ H_0(Y) & & \end{array} .$$

   Since both ε_* are now isomorphisms (Proposition 3.1.4), f_* also is.

7. Suppose C is a chain complex with an augmentation ε. Prove that the sequence

$$\ldots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \ldots \longrightarrow C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \longrightarrow \ldots$$

   is a chain complex C', and $H_n(\widetilde{C}) = H_n(C')$ for all $n \in \mathbb{Z}$. This gives another interpretation of reduced homology groups.

   Solution: Since $\varepsilon \circ \partial_1 = 0$, C' is a chain complex. For $n > 0$ or $n < 0$ it has the same groups of cycles and boundaries, so also the same homology as C. In other words

$$H_n(C') = H_n(C) = H_n(\widetilde{C}).$$

   If $n = 0$ the group of boundaries is the same as for C, but the group of cycles is $\operatorname{Ker} \varepsilon = \widetilde{C}_0$, so

$$H_0(C') = H_0(\widetilde{C}).$$

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.