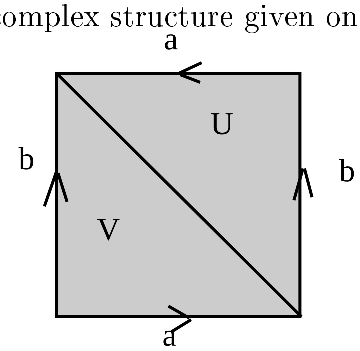
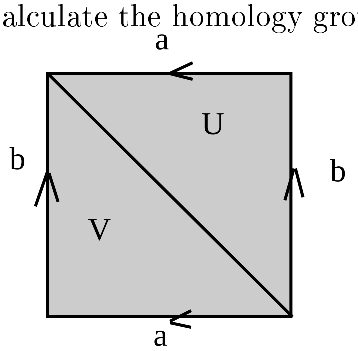
Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 5
10.10-15.10.2011

1. Consider S^1 as a polyhedron of a Δ-complex, generated by a single 1-simplex, whose vertices are identified. Calculate the simplicial homology of this complex. Verify that the result is the same as calculated in the lectures using the triangulation of S^1 as a boundary of a 2-simplex.

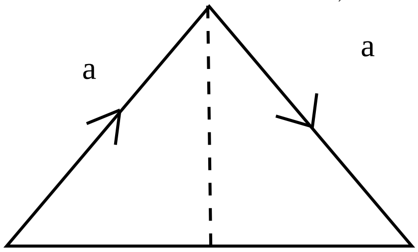
2. Calculate the homology groups of the Klein bottle, using the familiar Δ-complex structure given on the picture above.



3. Calculate the homology groups of the torus, using the Δ-complex structure



4. Calculate the homology groups of the Mobius band using the Δ-complex structure as in exercise 3.3, having one triangle with two sides indentified.



Check that you end up with the same result as in the lecture notes (where we used completely different triangulation of the Mobius band).

5. Find two short exact sequences of the form

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} C \xrightarrow{g} \mathbb{Z}_2 \longrightarrow 0,$$

$$0 \longrightarrow \mathbb{Z} \xrightarrow{f} C' \xrightarrow{g} \mathbb{Z}_2 \longrightarrow 0,$$

where C and C' are not isomorphic as groups.

6. Suppose H is a free abelian group, G is an abelian group and $f\colon G \to H$ is a surjective homomorphism. Prove that there is a homomorphism $f'\colon H \to G$ such that $f \circ f' = \mathrm{id}$.

7. Supppose a sequence

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

of chain complexes and chain mappings is exact.

Consider the sequence

$$\ldots \longrightarrow H_{n+1}(\overline{C}) \xrightarrow{\partial} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(\overline{C}) \xrightarrow{\partial} H_{n-1}(C') \longrightarrow \ldots$$

a) Prove that

$$\operatorname{Ker} \Delta \subset \operatorname{Im} g_*.$$

b) Prove the exactness at $H_n(C')$.

Here ∂ is a boundary operator, as constructed in lecture notes.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.