Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 4
03.10-09.10.2011

1. Let A be a set. For every $a \in A$ define $f_a \colon A \to \mathbb{Z}$ by

$$f_a(x) = \begin{cases} 1, & \text{if } x = a, \\ 0, & \text{otherwise} . \end{cases}$$

   Prove that the set $\{f_a \mid a \in A\}$ is a basis of the abelian group $\mathbb{Z}^{(A)}$.

2. Prove that a free abelian group G is torsion-free i.e. for every $g \in G$ and $n \in \mathbb{N}$ the equation

$$ng = 0$$

   is true if and only if $n = 0$ or $g = 0$.
   Conclude that $\mathbb{Z}_n$, $n \in \mathbb{N}$ or $\mathbb{Q}/\mathbb{Z}$ are not free abelian.

3. Suppose $A \subset \mathbb{Q}$ contains at least 2 points. Show that A is not independent. Conclude that $\mathbb{Q}$ is not a free abelian group, although it is torsion-free.

4. a) Let G be an abelian group and denote

$$2G = \{2g \mid g \in G\}.$$

   Prove that $2G$ is a subgroup of G and show that if $G \cong H$, then $G/2G \cong H/2H$.
   b) Suppose A and B are sets, A is finite. Prove that $\mathbb{Z}^{(A)} \cong \mathbb{Z}^{(B)}$ if and only if B is finite and has the same amount of elements as A (Hint: Prove that $\mathbb{Z}^{(A)}/2\mathbb{Z}^{(A)} = \mathbb{Z}_2^{(A)}$ and use a).)

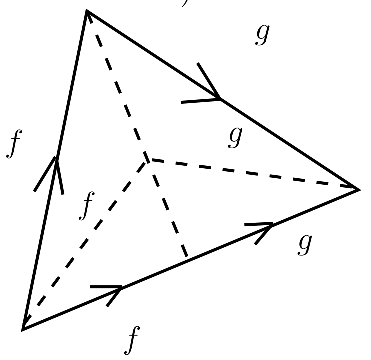
5. a)Let G be a free group on 3 free generators a, b, c. Show that $\{c-a, b-a, a\}$ is also a basis of G.
   b) Let G be a free group on 4 free generators a, b, c, d. Prove that the set $\{a+c+d, b-a+d, d\}$ is independent.

6. Suppose $\{\alpha, \beta\}$ is a basis of a group G. Prove that $\{\alpha \pm \beta, \beta\}$ is also a basis of G.

7. Suppose X is a topological space. Singular 1-simplices in X are mappings $f \colon I = [0,1] \to X$ and are also called **pathes** in X. If $f(0) = f(1)$ the path f is called **the loop**. Show that as an element of $C_1(X)$ the path f is a cycle if and only if it a loop.
   Suppose $f, g \colon I \to X$ are pathes and $g(0) = f(1)$. Then we can define their **product** $f \cdot g \colon I \to X$ by
   (continues on the other side)

$$(f \cdot g)(t) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq 1/2, \\ g(2t-1), & \text{if } 1/2 \leq t \leq 1. \end{cases}$$

Prove that in this case $f + g - f \cdot g$ is a boundary element in $C_1(X)$ by constucting the explicit 2-simplex in X, whose boundary is $f+g-f\cdot g$. (Hint: see the picture below.)



Conclude that if f and g are loops, then

$$[f] + [g] = [f \cdot g] \in H_1(X).$$

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.