Chapter 3

Singular homology

3.1 Zeroth homology group, path components and reduced homology

Suppose X is a topological space. Let $f\colon \Delta_n \to X$ be a singular simplex in X. Since Δ_n is path-connected, the image of f is also path-connected, hence contains in some path-component X_a of X. In particular as an element of $C_n(X)$ the chain f belongs to a subgroup $C_n(X_a)$.
It follows that the singular chain complex of X is completely determined by the singular chain complexes of its components. To formalize this in precise mathematical terms we need to make the following definition.

Suppose $(C_a, \partial_a)_{a\in\mathcal{A}}$ is a collection of chain complexes. We define their **direct sum** to be the chain complex $(C, \partial) = \oplus C_a$ defined by

$$C_n = \oplus (C_a)_n,$$

$$\partial (c_a)_{a\in A} = (\partial_a(c_a))_{a\in\mathcal{A}}.$$

For every $b \in A$ there are chain mappings $i_b\colon C_b \to C$ (inclusion) and $p_b\colon C \to C_b$ (projection).

Proposition 3.1.1. *Suppose X is a topological space and let $(X_a)_{a\in\mathcal{A}}$ be the set of all path-components of X. Then the chain inclusions $(i_\alpha)_\sharp\colon C(X_\alpha) \to C(X)$ induce a chain isomorphism*

$$(i_a)_{a\in\mathcal{A}}\colon \ \oplus_{a\in\mathcal{A}} C(X_a) \to C(X)$$

of chain complexes.

Proof. Exercise 3.1. □

This result translates to homology groups with the aid of the following general result.

Lemma 3.1.2. *Homology operation preserves direct sums of chain complexes. More precisely suppose the chain complex C is a direct sum of its subcomplexes $(C_a)_{a\in\mathcal{A}}$. Then inclusion mappings $i_a\colon C_a \to C$ induce a chain isomorphism*

$$((i_a)_*)_{a\in\mathcal{A}}\colon \oplus_{a\in\mathcal{A}} H_n(C_a) \to H_n(C)$$

for every $n \in \mathbb{Z}$.

Proof. Exercise 3.2a). ☐

Corollary 3.1.3. *Suppose X is a topological space let $(X_a)_{a\in\mathcal{A}}$ be the set of all path-components of X. Then the inclusions $i_a\colon X_a \to X$ induce an isomorphism*

$$((i_a)_*)_{a\in\mathcal{A}}\colon \oplus_{a\in\mathcal{A}} H_n(X_a) \to H_n(X)$$

for every $n \in \mathbb{N}$.

Proof. Exercise 3.2b). ☐

It follows that it is enough to study homology groups of the path-connected spaces.

Next we compute the 0-th homology of every space. For a topological space X define homomorphism of groups $\varepsilon\colon C_0(X) \to \mathbb{Z}$ by asserting

$$\varepsilon(\sigma) = 1$$

for every free generator $\sigma \in \mathrm{Sing}_0(X)$. Notice that the set $\mathrm{Sing}_0(X)$ can be identified with the set of points of X. Now $\partial_0 = 0$, since $C_{-1}(X) = 0$, so $\operatorname{Ker}\partial_0 = C_0(X)$ and hence

$$H_0(X) = C_0(X)/\operatorname{Im}\partial_1.$$

Suppose $f \in C_1(X)$. Then

$$\varepsilon(\partial_1(f)) = \varepsilon(f(1) - f(0)) = 1 - 1 = 0.$$

Since this is true for every free generator of $C_1(X)$, we have that

$$\varepsilon \circ \partial_1 = 0,$$

thusa ε induces the homomorphism

$$\varepsilon_*\colon H_0(X) \to \mathbb{Z}.$$

Moreover, if X is not an empty space, ε is a surjection, so also ε_* is a surjective mapping.

Proposition 3.1.4. *Suppose X is path-connected and non-empty. Then $\varepsilon_*\colon H_0(X) \to \mathbb{Z}$ is an isomorphism.*

Proof. Since we already observed that ε is surjective, it is enough to prove that $\operatorname{Ker}\varepsilon = \operatorname{Im}\partial_1$. The result then follows from the isomorphism theorem of the group theory.
The inlusion $\operatorname{Im}\partial_1 \subset \operatorname{Ker}\varepsilon$ is already proved above for all spaces. Conversly suppose that

$$c = \sum_{i=1}^{k} n_i x_i \in \operatorname{Ker}\varepsilon,$$

for some $k \in \mathbb{N}$, $n_i \in \mathbb{Z}$, $x_i \in X$. Then

$$\sum_{i=1}^{k} n_i = \varepsilon(c) = 0.$$

Fix a point $x \in X$ (for example x_0). Since X is path-connected, for every $i = 1, \ldots, k$ there is a path $f_i\colon I \to X$ from x to x_i, i.e. $f(0) = x, f(1) = x_i$. Now

$$d = \sum_{i=1}^{k} n_i f_i \in C_1(X) \text{ and}$$

$$\partial_1(d) = \sum_{i=1}^{k} n_i(x_i - x) = \sum_{i=1}^{k} n_i x_i - \Big(\sum_{i=1}^{k} n_i\Big)x = \sum_{i=1}^{k} n_i x_i = c.$$

Hence $c \in \operatorname{Im}\partial_1$ and we are done. □

Since $1 \in \mathbb{Z}$ is a generator of the free group $\mathbb{Z}$ and $\varepsilon_*[x] = 1$ for every $x \in X$, it follows that as a generator of $H_0(X)$ for the path-connected space X we can take a homology class $[x]$ of any fixed point $x \in X$ (which is a 0-simplex in X).

Corollary 3.1.5. *Suppose X is a topological space. Then $H_0(X)$ is a free abelian group on the set of path components of X.*
If $H_0(X) = \mathbb{Z}^n$ for $n \in \mathbb{N}$, then X has exactly n components. In particular X is path-connected if and only $H_0(X) \cong \mathbb{Z}$.

Proof. The first assertion follows from the previous proposition and Corollary 3.1.3. The second follows from the the fact that $\mathbb{Z}^{(A)} = \mathbb{Z}^n$ if and only if A has n elements (exercise (2.9). □

Notice that in general as free generators of $H_0(X) \cong \mathbb{Z}^{(A)}$ we can take a collection of points $(x_a)_{a\in A}$ (or their homology classes to be precise), where exactly one point is chosen from every path-component of X.

Consider a chain complex C' with $C'_n = 0$ for $n \neq 0$, $C_0 = \mathbb{Z}$ and all boundary operators zero. We will denote this complex simply by $\mathbb{Z}$, slightly abusing the notation. Obviously $H_n(\mathbb{Z}) = 0$ for $n \neq 0$ and $H_0(\mathbb{Z}) = \mathbb{Z}$.
Let C be an arbitrary chain complex. A chain mapping $\varepsilon\colon C \to \mathbb{Z}$ reduces to a single homomorphism $\varepsilon\colon C_0 \to \mathbb{Z}$ subject to a single condition

$$\varepsilon \circ \partial_1 = 0$$

since the diagram

$$\begin{array}{ccc} C_1 & \longrightarrow & 0 \\ \downarrow{\scriptstyle \partial_1} & & \downarrow \\ C_0 & \xrightarrow{\varepsilon} & \mathbb{Z} \end{array}$$

must be commutative.
Such a homomorphism $\varepsilon\colon C_0 \to \mathbb{Z}$ is called **an augmentation** of the complex C if C is non-negative and ε is also surjective. The pair (C, ε) is then called **an augmented chain complex**. Above we have constructed a canonical natural augmentation of the complex $C(X)$ for every non-empty topological space X (notice that for empty space the constructed mapping is not surjective, hence not an augmentation).

Since $\varepsilon\colon C \to \mathbb{Z}$ is a chain mapping, its kernel $\widetilde{C} = \operatorname{Ker} \varepsilon$ is a chain complex, a subcomplex of C (exercise). Clearly $\widetilde{C}_n = C_n$ for $n \neq 0$ and since ε is surjective in all dimensions, we have an exact short sequence

$$0 \longrightarrow \widetilde{C} \xrightarrow{i} C \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

of chain complex and chain mappings. From the corresponding long exact homology sequence we obtain for $n > 0$ the exact sequence

$$H_{n+1}(\mathbb{Z}) = 0 \longrightarrow H_n(\widetilde{C}) \xrightarrow{i} H_n(C) \longrightarrow H_n(\mathbb{Z}) = 0$$

and for $n = 0$ the exact sequence

$$H_1(\mathbb{Z}) = 0 \longrightarrow H_0(\widetilde{C}) \xrightarrow{i} H_0(C) \xrightarrow{\varepsilon_*} \mathbb{Z} = H_0(\mathbb{Z}) \longrightarrow 0$$

Since $\mathbb{Z}$ is free, it follows from the lemma 2.2.12 that last sequence splits. Hence it follows that

$$H_n(\widetilde{C}) = H_n(C) \text{ for } n > 0$$

$$H_0(C) = H_0(\widetilde{C}) \oplus \mathbb{Z} \text{ and}$$
$$H_0(\widetilde{C}) = \operatorname{Ker} \varepsilon_*.$$

In particular these considerations apply to the non-negative chain complex $C(X)$ for non-empty topological space X, with augmentation ε defined above. The homology groups $H_n(\widetilde{C(X)})$ of the chain complex $\widetilde{C(X)}$ are called **reduced singular homology groups** of the space X and are denoted $\widetilde{H}_n(X)$. It follows that

$$\widetilde{H}_n(X) = H_n(X) \text{ if } n > 0,$$

$$H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z} \text{ and}$$
$$\widetilde{H}_0(X) = \operatorname{Ker} \varepsilon_*.$$

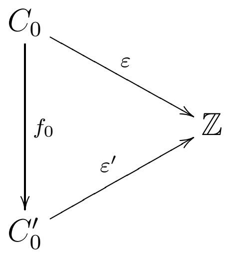
It can be proved that $\widetilde{H}_0(X)$ is also free abelian (it is trivial, if you know that any subgroup of a free group is free). For our purposes the following result will suffice.

Proposition 3.1.6. *Suppose X is a non-empty topological space. Then $\widetilde{H}_0(X) = 0$ if and only X is path-connected.*

Proof. If X is path-connected, ε_* is an isomorphism, in particular $\widetilde{H}_0(X) = \operatorname{Ker} \varepsilon_* = 0$. Conversly if $\widetilde{H}_0(X) = 0$, then $H_0(X) = \widetilde{H}_0(X) \oplus \mathbb{Z} \cong \mathbb{Z}$, so X is path-connected by the corollary 3.1.5. □

Notice that for empty space reduced homology groups are not defined.

Suppose (C, ε) and (C', ε') are augumented chain complexes. The chain mapping of augmented complexes $f\colon (C, \varepsilon) \to (C', \varepsilon')$ is a chain mapping that commutes with augmentation. Again this definition reduces to f being a chain mapping for which the diagram



commutes. It follows that f maps $\widetilde{C}$ to $\widetilde{C'}$, hence there is an induced mapping $f_*\colon H_0(\widetilde{C}) \to H_0(\widetilde{C'})$.

Suppose

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0$$

is a short exact sequence of chain complexes, where C' and C are augmented and f preserves augmentation. Then the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \widetilde{C'}, & \xrightarrow{\widetilde{f}} & \widetilde{C} & \xrightarrow{g|} & \overline{C} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C' & \xrightarrow{f} & C & \xrightarrow{g} & \overline{C} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varepsilon'} & & \downarrow{\scriptstyle \varepsilon} & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{id} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

is commutative. Moreover all columns are exact, as well as the middle and bottom rows. It turns out that this implies that also the upper row is exact. This follows from more general result.

Lemma 3.1.7. *Suppose*

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & A_{n-1} & \longrightarrow & A_n & \longrightarrow & A_{n+1} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & B_{n-1} & \longrightarrow & B_n & \longrightarrow & B_{n+1} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & C_{n-1} & \longrightarrow & C_n & \longrightarrow & C_{n+1} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

is a commutative diagram of abelian groups and homomorphisms. Assume also that all columns are exact and the middle row is exact. Then the upper row is exact if and only if lower row is exact.

Proof. Exercise 3.4. ☐

Hence it follows that in the situation above the sequence

$$0 \longrightarrow \widetilde{C'} \xrightarrow{f|} \widetilde{C} \xrightarrow{g|} \overline{C} \longrightarrow 0$$

is the short exact sequence of chain complexes and chain mappings. From the theorem 2.2.5 it follows that there is a long exact sequence in homology

$$\ldots \longrightarrow H_{n+1}(\overline{C}) \xrightarrow{\partial} H_n(C') \xrightarrow{f_*} H_n(C) \xrightarrow{g_*} H_n(\overline{C}) \xrightarrow{\partial} H_{n-1}(C') \longrightarrow \ldots$$

$$\ldots \longrightarrow H_1(\overline{C}) \xrightarrow{\partial} H_0(\widetilde{C'}) \xrightarrow{f_*} H_0(\widetilde{C}) \xrightarrow{g_*} H_0(\overline{C}) \longrightarrow 0 \qquad ,$$

called **the reduced long homology sequence** of the original exact sequence

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} \overline{C} \longrightarrow 0.$$

Let us apply this results to the singular homology. Suppose (X, A) is a topological pair and $A \neq \emptyset$. Then there is exact sequence

$$0 \longrightarrow C(A) \xrightarrow{i_\sharp} C(X) \xrightarrow{j_\sharp} C(X, A) \longrightarrow 0,$$

where $C(A)$ and $C(X)$ are augmented. Moreover it is easy to check that inclusion $i_\sharp$ commutes with augmentation. Hence by the results above we obtain **the reduced long singular homology sequence**

$$\ldots \longrightarrow H_{n+1}(X, A) \xrightarrow{\partial} H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \longrightarrow \ldots$$

$$\ldots \longrightarrow H_1(X, A) \xrightarrow{\partial} \widetilde{H}_0(A) \xrightarrow{i_*} \widetilde{H}_0(X) \xrightarrow{j_*} H_0(X, A) \longrightarrow 0 \qquad .$$

Notice that the chain mapping $f_\sharp \colon C(X) \to C(Y)$ induced by any continuous mapping $f \colon X \to Y$ preserves augmentation, hence induces homomorphism $f_* \colon \widetilde{H}_0(X) \to \widetilde{H}_0(Y)$.

Reduced homology sequence is natural with respect to chain mappings that preserve augmentation. In particular in case of singular homology it is natural with respect to the mappings induced by continuous mappings.

In the end of this section let us compute the singular homology of a singleton space $X = \{x\}$. It is clear that for every $n \in \mathbb{N}$ there is a single mapping $\sigma_n \colon \Delta_n \to X$, so $C_n(X)$ is a free abelian group generated by a single element σ_n, in particular isomorphic to $\mathbb{Z}$. Clearly $\partial_i(\sigma_n) = \sigma_{n-1}$ for all $n \geq 1$ and $i = 0\dots, n$, so

$$\partial(\sigma_n) = \sum_{i=0}^{n}(-1)^n \sigma_{n-1} = \begin{cases} \sigma_{n-1}, \text{ if } n \text{ is even} \\ 0, \text{ if } n \text{ is odd} \end{cases}$$

Hence the complex $C(X)$ is a sequence

$$\dots \longrightarrow \mathbb{Z} = C_{2n+1} \xrightarrow{0} \mathbb{Z} = C_{2n} \xrightarrow{\text{id}} \mathbb{Z} = C_{2n-1} \longrightarrow \dots \longrightarrow \mathbb{Z} = C_1 \xrightarrow{0} \mathbb{Z} = C_0 \xrightarrow{0} 0.$$

Easy computations and proposition 3.1.6 show that

$$H_n(X) = 0 \text{ for } n > 0$$

$$H_0(X) = \mathbb{Z}$$
$$\widetilde{H}_n(X) = 0.$$

3.2 Homotopy axiom

Homotopy axioms asserts that homotopic mappings induce the same homomorphism in the homology. Precisely put

Proposition 3.2.1. *Suppose $f, g\colon (X, A) \to (Y, B)$ are homotopic as mappings of pairs i.e. there exists a mapping $F\colon (X \times I, A \times I) \to (Y, B)$ of pairs for which*

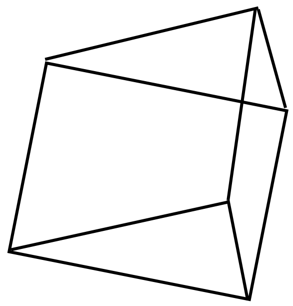
$$F(x, 0) = f(x),$$
$$F(x, 1) = g(x)$$

for all $x \in X$. Then

$$f_* = g_*\colon H_n(X, A) \to H_n(Y, B), n \in \mathbb{N}.$$

In the absolute case the same is true for reduced groups

Suppose $f, g\colon (X, A) \to (Y, B)$ and $F\colon (X \times I, A \times I) \to (Y, B)$ is a homotopy between f and g. For a singular simplex $\sigma\colon \Delta_n \to X$ we have a homotopy $F \circ (\sigma \times \text{id})\colon \Delta_n \times I \to Y$ between $f_\sharp(\sigma)$ and $g_\sharp(\sigma)$. Now $\Delta \times I$ is not a simplex, but it is **a prism**, which is a polyhedron, i.e. can be triangulated, so that the bottom and the top (which are both n-simplices) preserve their simplicial structure.



To be precise bottom and the top are both simplices with vertices $v_i = (e_i, 0)$ and v_i', $i \in \{0, \ldots, n\}$. It can be verified that the ordered $n+1$-simplices $[v_0, \ldots, v_i, v_i', \ldots, v_n']$, $i = 0, \ldots, n$ are really geometrical simplices and form a simplicial complex, which is a triangulation of $\Delta_n \times I$. We will not prove this and leave this claim as an exercise for the interested reader, since we really don't need this fact, it merely provides us with the motivation and idea for the proof, which works on the formal algebraic level nevertheless.

Inspired by this for every $\sigma \in \mathrm{Sing}_n(X)$ we define $\sigma_i \colon \Delta_{n+1} \to X \times I$ by restricting $\sigma \times \mathrm{id}$ on the subset $[v_0, \ldots, v_i, v_i', \ldots, v_n']$. To be more precise define $\alpha_i = \alpha_i^n \colon \Delta_{n+1} \to \Delta_n \times I$ to be the unique convex mapping that maps vertices $(e_0, \ldots, e_{n+1})$ to the points $(v_0, \ldots, v_i, v_i', \ldots, v_n')$ in that order. Such a mapping clearly exists, since the prism $\Delta_n \times I$ is a convex subset of $\mathbb{R}^{n+1}$. Then define

$$\sigma_i = (\sigma \times \mathrm{id}) \circ \alpha_i.$$

Next we define so-called **prism operator** $P_n \colon C_n(X) \to C_{n+1}(Y)$ for every $n \in \mathbb{N}$. On the generators we assert

$$P_n(\sigma) = \sum_{i=0}^{n} (-1)^i (F \circ \sigma_i).$$

Claim: For all $n \in \mathbb{N}$

$$\partial_{n+1} P_n = f_\sharp - g_\sharp - P_{n-1}\partial_n.$$

Geometrically we can think of the left side of this equation representing the whole boundary of the prism (the top, the bottom, and horizontal sides), while the right side is the signed sum of the bottom, top and all horizontal sides.

Proof of the claim:

$$\partial_{n+1} P_n(\sigma) = \sum_{j=0}^{n+1} \sum_{i=0}^{n} (-1)^{i+j} (F \circ \sigma_i \circ \varepsilon_j) =$$

$$\sum_{j\le i}(-1)^{i+j}F\circ(\sigma\times \mathrm{id})[v_0,\dots,\hat{v}_j,\dots,v_i,v'_i,\dots,v'_n]+$$

$$\sum_{j\ge i}(-1)^{i+j+1}F\circ(\sigma\times \mathrm{id})[v_0,\dots,v_i,v'_i,\dots,\hat{v}_j,\dots,v'_n],$$

where we switched index j to $j-1$ in the second sum.
Now the terms with $i=j$ cancel out, except for the first term $F\circ(\sigma\times \mathrm{id})[v'_0,\dots,v'_n]$, which is $g\circ\sigma=g_\sharp(\sigma)$ and the last term $-F\circ(\sigma\times\mathrm{id})[v_0,\dots,v_n]$, which is $-f_\sharp(\sigma)$. Let us look closely why this happens. For every $i=0,\dots n$ let

$$x_i=[v_0,\dots,\hat{v}_i,v'_i,\dots,v'_n],$$

$$y_i=[v_0,\dots,v_i,\hat{v'_i},\dots,v'_n].$$

Then $x_{i+1}=y_i$ for all $i=0,\dots,n-1$. Moreover every x_i occurs in the sum above with plus-sign, while every y_i occurs with the minus-sign. Hence x_1 and y_0 cancel each other out, x_2 and y_1 also cancel each other and so on, with the last pair being x_n, y_{n-1}.

What about $P\partial_n$? Now

$$(\sigma\varepsilon^j)_i=(\sigma\times\mathrm{id})\circ(\varepsilon^j\times\mathrm{id})\circ\alpha_i^{n-1},$$

where

$$(\varepsilon^j\times\mathrm{id})\circ\alpha_i^{n-1}=\begin{cases}[v_0,\dots,\hat{v}_j,\dots,v_{i+1},v'_{i+1},\dots,v'_n], & \text{if } j\le i,\\ [v_0,\dots,v_i,v'_i,\dots,\hat{v}_j,\dots,v'_n], & \text{if } j>i.\end{cases}$$

Hence

$$P\partial_n(\sigma)=\sum_{j\le i<n}(-1)^i(-1)^j(F\circ\sigma\times\mathrm{id})[v_0,\dots,\hat{v}_j,\dots,v_{i+1},v'_{i+1},\dots,v'_n]+$$

$$\sum_{i<j\le n}(-1)^i(-1)^j(F\circ\sigma\times\mathrm{id})[v_0,\dots,v_i,v'_i,\dots,\hat{v}_j,v'_n]=$$

$$=\sum_{j<i\le n}(-1)^{i+j+1}(F\circ\sigma\times\mathrm{id})[v_0,\dots,\hat{v}_j,\dots,v_i,v'_i,\dots,v'_n]+$$

$$+\sum_{i<j}(-1)^{i+j}(F\circ\sigma\times\mathrm{id})[v_0,\dots,v_i,v'_i,\dots,\hat{v}_j,v'_n].$$

Thus

$$\partial_{n+1}P_n(\sigma)=g_\sharp(\sigma)-f_\sharp(\sigma)-P\partial_n.$$

Now if $\sigma \in C_n(A)$, it follows by definitions that $\sigma_i \in C_{n+1}(A \times I)$, hence $F \circ \sigma_i \subset C_{n+1}(B)$. In other words P_n maps $C_n(A)$ into $C_{n+1}(B)$, hence induces a homomorphism $\overline{P}_n \colon C_n(X, A) \to C_{n+1}(Y, B)$. The formula $\partial_{n+1} P_n = f_\sharp - g_\sharp - P_{n-1}\partial_n$. implies the formula

$$\overline{\partial}_{n+1}\overline{P}_n = f_\sharp - g_\sharp - \overline{P}_{n-1}\overline{\partial}_n$$

in quotient groups.

The mappings $\overline{P}_n$ is an example of what is generally known as the chain homotopy.

Definition 3.2.2. *Suppose $f, g \colon C \to C'$ are chain mappings between chain complexes. The collection $H = (H_n)_{n \in \mathbb{N}}$ of homomorphisms $H_n \colon C_n \to C'_{n+1}$ is called* **a chain homotopy** *between f and g if*

$$\partial'_{n+1} H_n + H_{n-1}\partial_n = f_n - g_n.$$

In this case we say that f and g are **chain homotopic***.*

We have shown that $f_\sharp, g_\sharp \colon C(X, A) \to C(Y, B)$ are chain homotopic chain mappings.
The homotopy axiom follows now from the following general result.

Lemma 3.2.3. *Suppose $f, g \colon C \to C'$ are chain homotopic. Then*

$$f_* = g_* \colon H_n(C) \to H_n(C')$$

for all $n \in \mathbb{N}$.

Proof. Suppose $c \in Z_n(C)$ is a cycle. Then

$$f(c) - g(c) = \partial' H(c) + H\partial(c) = \partial' H(c),$$

so $[f(c)] = [g(c)]$ in homology. □

The claim for the reduced case follows easily, since $f_* \colon \tilde{H}_0(X) \to \tilde{H}_0(Y)$ is just the restriction of $f_* \colon H_0(X) \to H_0(Y)$ to a subgroup.

A mapping $f \colon (X, A) \to (Y, B)$ is called **a homotopy equivalence** if there exists $g \colon (Y, B) \to (X, A)$ such that $f \circ g \simeq \mathrm{id}_{(Y,B)}$, $g \circ f \simeq \mathrm{id}_{(X,A)}$ (as mappings of pairs).
Mapping g is called **a homotopy inverse** of f. The pairs (X, A) and (Y, B) are said to have **the same homotopy type** if there exists homotopy equivalence $f \colon (X, A) \to (Y, B)$.
It is clear that homeomorphic pairs has the same homotopy type.

Example 3.2.4. *1)* $\mathbb{R}^n$ *has the same homotopy type as* $\overline{B}^n$, B^n *or a singleton* $\{a\}$.
In fact every non-empty convex subset C *of a finite dimensional vector space have the homotopy type of a singleton space. Moreover if* $x \in C$ *the pair* (C, x) *has the same homotopy type as the pair* $(\{x\}, \{x\})$ *(Exercise).*
2) $\mathbb{R}^n \setminus \{0\}$ *has the same homotopy type as* S^{n-1} *or a punctured ball* $\overline{B}^n \setminus \{0\}$. *(Exercise).*
3) Mobius band has the same homotopy type as S^1 *(Exercise).*

Corollary 3.2.5. *Suppose* $f\colon (X, A) \to (Y, B)$ *is a homotopy equivalence. Then*

$$f_*\colon H_n(X, A) \to H_n(Y, B)$$

is an isomorphism. The same is true for reduced groups in the absolute case. In particular spaces of the same homotopy type have the same homology groups.

Proof. Suppose $g\colon (Y, B) \to (X, A)$ is a homotopy inverse of f. Then $g \circ f \simeq \mathrm{id}$ as mappings $(X, A) \to (X, A)$, so by homotopy axiom

$$g_* \circ f_* = (g \circ f)_* = \mathrm{id}\colon H_n(X, A) \to H_n(X, A) \text{ for all } n \in \mathbb{N}.$$

Similarly $f_* \circ g_* = \mathrm{id}\colon H_n(Y, B) \to H_n(Y, B)$. Hence g_* is the inverse of f_*. □

Recall that a topological space X is called contractible, if identity mapping $\mathrm{id}\colon X \to X$ is homotopic to a constant mapping $x_0\colon X \to X$ for some $x_0 \in X$. This means precisely that there exists a mapping $H\colon X \times I \to X$ such that $H(x, 0) = x$, $H(x, 1) = x_0$ for all $x \in X$.
If this homotopy is stable at x_0 i.e. $H(x_0, t) = x_0$ for all $t \in I$, we say that the pair (X, x_0) is contractible.

Lemma 3.2.6. *The space* X *is contractible if and only if it has the same homotopy type as a singleton space* $\{x\}$.
Similarly the pair (X, x) *is contractible if and only if it has the same homotopy type as the pair* $(\{x\}, \{x\})$.
Every contractible space is path-connected.

Proof. Exercise 3.12. □

Example 3.2.7. *As we already noticed* $\mathbb{R}^n$, $\overline{B}^n$ *and* B^n *are contractible. More generally any convex subset* C *of a finite dimensional vector space* V *is contractible.*

Example 3.2.8. *Consider the so-called "topological comb"-space X defined as*

$$X = \bigcup_{n\in\mathbb{N}_+} \{1/n\} \times I \cup \{0\} \times I \cup I \times \{0\}.$$

Then X is contractible. Let $x_0 = (0,1)$. Then the pair (X, x_0) is **not** *contractible. Proofs are left as an exercise.*

Since spaces with the same homotopy type have the same homology and the homology of the singleton space is already calculated, we obtain the following result.

Corollary 3.2.9. *Suppose X is a contractible space. Then*

$$H_n(X) = 0 \text{ for } n > 0,$$

$$H_0(X) \cong \mathbb{Z},$$

$$\widetilde{H}_0(X) = 0.$$

In particular this is true for $X = \mathbb{R}^n, \Delta_n, \overline{B}^n, B^n$ for all $n \in \mathbb{N}$.

3.3 Excision

Excision property is perhaps the most powerful and important property of the singular homology. It makes homology groups highly computible and "well-behaved".
Formally **Excision axiom** is the following statement

Theorem 3.3.1. *Suppose $A \subset U \subset X$. If $\overline{A} \subset \operatorname{int} U$, then the inclusion $i\colon (X \setminus A, U \setminus A) \to (X, U)$ induces an isomorphism*

$$i_*\colon H_n(X \setminus A, U \setminus A) \to H_n(X, U)$$

for all $n \in \mathbb{Z}$.

In other words under the assumptions of the theorem you can "cut out " or "excite" the set A from the pair (X, U) without altering the homology.
Before proving this theorem let us give an example of its application, which will illuminate its importance and the way this property is applied in the practice.
Suppose one wants to calculate the homology of a sphere S^n. It is enough to compute reduced homology groups. Let

$$U = S^n \setminus \{e_{n+1}\} \subset \mathbb{R}^{n+1}.$$

U is homeomorphic to $\mathbb{R}^n$ (stereographic projection from the "north pole", exercise 3.15a or Topology II), in particular contractible, so its reduced homology groups are trivial. From the long exact reduced homology sequence

$$\tilde{H}_m(U) = 0 \longrightarrow \tilde{H}_m(S^n) \xrightarrow{j_*} H_m(S^n, U) \xrightarrow{\partial} \tilde{H}_{m-1}(U) = 0$$

we see that $j_*\colon \tilde{H}_m(S^n) \cong H_m(S^n, U)$ is an isomorphism, so it is enough to compute $H_m(S^n, U)$. Let

$$A = \{x = (x_1, \ldots, x_{n+1}) \in S^n \mid x_{n+1} < 0\}.$$

Then $\overline{A} = \{x \in S^n \mid x \leq 0\} \subset U = \operatorname{int} U$, so excision axiom implies that

$$H_m(S^n, U) \cong H_m(S^n \setminus A, U \setminus A).$$

Now $S^n \setminus A$ is a closed upper hemisphere $\{x \in S^n | x_{n+1} \geq 0\}$, which is homeomorphic to the closed ball $\overline{B}^n$ (exercise 3.15b). Under this homeomorphism $U \setminus A$ corresponds to the punctured ball $\overline{B}^n \setminus \{0\}$. Hence

$$H_m(S^n \setminus A, U \setminus A) \cong H_m(\overline{B}^n, \overline{B}^n \setminus \{0\}).$$

The inclusion of pairs $(\overline{B}^n, S^{n-1}) \to (\overline{B}^n, \overline{B}^n \setminus \{0\})$ is a homotopy equivalence of pairs (exercise 3.13), so it induces an isomorphism

$$H_m(\overline{B}^n, \overline{B}^n \setminus \{0\}) \cong H_m(\overline{B}^n, S^{n-1}).$$

On the other hand $\overline{B}^n$ is contractible, so its reduced homology groups are trivial. From the long exact reduced homology sequence of the pair $(\overline{B}^n, S^{n-1})$ we see that

$$H_m(\overline{B}^n, S^{n-1}) \cong \widetilde{H}_{m-1}(S^{n-1}).$$

Thus we have proved that for all $m \in \mathbb{N}$

$$\widetilde{H}_m(S^n) \cong \widetilde{H}_{m-1}(S^{n-1}).$$

Noticed that if we would use ordinary groups instead of reduced, we would have to deal with exceptional case $m = 0, 1$ and the computations would be more involved, complicated and un-symmetric. This is a typical illustration of the convenience of reduced groups.

Now we can proceed by induction. We already know the reduced homology groups of S^0 (exercise 3.7) which are

$$\widetilde{H}_m(S^0) = 0 \text{ for } m \neq 0,$$

$$\widetilde{H}_0(S^0) \cong \mathbb{Z}.$$

Hence the previous computations imply by induction the following important result (and the first interesting example of non-trivial homology groups).

Theorem 3.3.2. *Singular homology groups of the sphere S^n, $n \in \mathbb{N}$ are the following.*

$$H_m(S^n) = \begin{cases} \mathbb{Z}, & \text{if } m = n \neq 0 \text{ or } m = 0, n \neq 0 \\ 0, & \text{if } m \neq n, 0 \\ \mathbb{Z} \oplus \mathbb{Z}, & \text{if } m = n = 0. \end{cases}.$$

Corollary 3.3.3. *If $n \neq m$ spheres S^n and S^m don't have the same homotopy type. In particular they are not homeomorphic. Any sphere S^n is not contractible.*

Also we immediately obtain the promised classical result.

Corollary 3.3.4. *Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^m$ are not homeomorphic if $n \neq m$.*

Proof. Suppose $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a homeomorphism. By composing it with a translation, if necessary, we may assume that $f(0) = 0$. Hence f induces a homeomorphism $\mathbb{R}^n \setminus \{0\} \to \mathbb{R}^m \setminus \{0\}$. In particular these spaces have the same homotopy type. But $\mathbb{R}^n \setminus \{0\}$ has the same homotopy type as S^{n-1}, so we obtain that S^{n-1} and S^{m-1} have the same homotopy type. This contradicts previous corollary. □

This result can be slightly generalized - in the exercise 3.16 you are asked to prove that if $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are both open, non-empty and there is a homeomorphism $f\colon U \to V$, then $n = m$.

Another corollary is that S^{n-1} is nor a retract of $\overline{B}^n$. Recall that a continuous mapping $p\colon X \to A$ is called a **retraction** if A is a subspace of X and $p|A = \mathrm{id}_A$. In other words if $i\colon A \to X$ denotes the inclusion, p is retraction if and only if $p \circ i = \mathrm{id}_A$. If $p\colon X \to A$ is a retraction, we say that A is a **retract** of X.

Corollary 3.3.5. *S^{n-1} is not a retract of $\overline{B}^n$.*

Proof. Suppose $p\colon \overline{B}^n \to S^{n-1}$ is such that $p \circ i = \mathrm{id}_A$. This implies in particular that

$$p_* \circ i_* = \mathrm{id}\colon \widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(S^{n-1}).$$

It follows that $i_*\colon \widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(\overline{B}^n)$ is an injection. This is however not possible, since $\widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z} \neq 0$, while $\widetilde{H}_{n-1}(\overline{B}^n) = 0$. □

It remains to prove the excision property. The proof is rather long and tedious. In fact we prove more general result stated below in the theorem 3.3.6.

Suppose $\mathcal{U} = (U_i)_{i \in I}$ is a covering of a topological space X. Denote by $C_n^{\mathcal{U}}(X)$ a free subgroup of $C_n(X)$ generated by those simplices $\sigma\colon \Delta_n \to X$ with the property $\sigma(\Delta_n) \subset U_i$ for some $i \in I$. It is elementary to check that the collection of these subgroups form a chain subcomplex $C^{\mathcal{U}}(X)$ of $C(X)$. The homology groups of this chain complex are denoted $H_n^{\mathcal{U}}(X)$.

Theorem 3.3.6. *Suppose $\mathcal{U}$ is a covering of X with the property that the collection $\{\operatorname{int} U | U \in \mathcal{U}\}$ is also a covering of X. Then the inclusion mapping $i\colon C^{\mathcal{U}}(X) \to C(X)$ induces isomorphisms in homology for every $n \in \mathbb{N}$,*

$$i_*\colon H_n^{\mathcal{U}}(X) \cong H_n(X).$$

Let us first check that the this theorem implies excision property. Suppose $A \subset U \subset X$ is such that $\overline{A} \subset \operatorname{int} U$. Denote $V = X \setminus A$, $\mathcal{U} = \{U, V\}$. Then

$$\operatorname{int} V = \operatorname{int}(X \setminus A) = X \setminus \overline{A}, \text{ thus}$$

$$\operatorname{int} V \cup \operatorname{int} U = X,$$

and the covering $\mathcal{U}$ satisfies conditions of the theorem 3.3.6. Hence $i\colon C(V) + C(U) \subset C(X)$ induces isomorphisms in homology. Consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(U) & \longrightarrow & C(V)+C(U) & \longrightarrow & (C(V)+C(U))/C(U) & \longrightarrow & 0 \\
 & & \downarrow \text{id} & & \downarrow i & & \downarrow & & \\
0 & \longrightarrow & C(U) & \longrightarrow & C(X) & \longrightarrow & C(X,U) & \longrightarrow & 0
\end{array}$$

with exact rows. By the naturality of the long homology sequence we obtain a commutative diagram

$$\begin{array}{ccccccccc}
H_n(U) & \longrightarrow & H_n^{\mathcal{U}}(X) & \longrightarrow & H_n((C(V)+C(U))/C(U)) & \longrightarrow & H_{n-1}(U) & \longrightarrow & H_{n-1}^{\mathcal{U}}(X) \\
\downarrow \text{id} & & \downarrow i_* & & \downarrow & & \downarrow \text{id} & & \downarrow i_* \\
H_n(U) & \longrightarrow & H_n(X) & \longrightarrow & H_n(X,U) & \longrightarrow & H_{n-1}(U) & \longrightarrow & H_{n-1}(X).
\end{array}$$

with exact rows. Now i_* is isomorphism for all $n \in \mathbb{N}$, and the identity mapping $\text{id}\colon H_n(U) \to H_n(U)$ is trivially isomorphism. By the five-lemma it follows that also induced mapping $H_n((C(V) + C(U))/C(U)) \to H_n(X, U)$ is an isomorphism.

Now consider a mapping $j\colon C(V)/C(U\cap V) \to (C(V)+C(U))/C(U)$ induced by the inclusion $C(V) \hookrightarrow C(V)+C(U)$. Notice that $C(U\cap V) = C(U)\cap C(V)$. By **the second isomorphism theorem of the group theory** (look up Algebra I, if you don't remember it or prove yourself) j is a (chain) isomorphism. In particular it induces isomorphisms in homology groups.
Collecting all these data together gives us the isomorphism $H_n(V, V \cap U) \cong H_n(X, U)$ induced by the inclusion, for all $n \in \mathbb{N}$. Since $V = X \setminus A$ and $V \cap U = U \setminus V$, this is precisely the excision axiom.

Hence it remains to prove the theorem 3.3.6. We prove it by showing that $i\colon C^{\mathcal{U}}(X) \to C(X)$ is a **chain homotopy equivalence** i.e. there is a chain mapping $p\colon C(X) \to C^{\mathcal{U}}(X)$ such that $p \circ i$ and $i \circ p$ are chain homotopic to the identity mapping. Since chain homotopic mappings induce the same homorphisms in homology (lemma 3.2.3), it follows that

$$p_* \circ i_* = \mathrm{id}, i_* \circ p_* = \mathrm{id},$$

so i_* is indeed an isomorphism.

The construction of j and homotopies involved is done in several steps. Suppose V is a finite-dimensional vector space and $D \subset V$ is a convex subset. Denote by $LC_n(D)$ a subgroup of $C_n(D)$ generated by singular n-simplices $f\colon \Delta_n \to D$, which are affine mappings. Notice that such a mapping is uniquely determined by $(n+1)$-tuple $\{f(e_0), \ldots, f(e_n)\} \in D^{n+1}$. Since D is convex, conversely any element $(d_0, \ldots, d_n)$ of D^{n+1} defines a unique affine mapping $f\colon \Delta_n \to D$ with $f(e_i) = d_i$. Thus we might as well think of LD_n as a free group generated by the set D^{n+1}. Since

$$\partial(d_0, \ldots, d_n) = \sum_{i=0}^{n} (d_0, \ldots, \widehat{d_i}, \ldots d_n)$$

subgroups $LC_n(D)$ define a chain subcomplex $LC(D)$ of $C(D)$. Notice that $LC_0(D) = C_0(D)$, so consequently the chain complex $LC(D)$ had an augmentation $\varepsilon\colon LC_0(D) \to \mathbb{Z}$ defined by $\varepsilon(d_0) = 1$. Fix a point $b \in D$ and define for every $n \in \mathbb{N}$ a homomorphism $b\colon LC_n(D) \to LC_{n+1}(D)$ by

$$b(d_0, \ldots, d_n) = (b, d_0, \ldots, d_n).$$

Straightforward calculation shows (exercise 3.18) that

$$(\partial_{n+1} b + b\partial_n)(x) = \begin{cases} x, \text{ if } n > 0, \\ x - \varepsilon(x) b, \text{ if } n = 0. \end{cases}$$

It follows that the restriction of b to $\widetilde{LC}_n$ is a chain homotopy between the identity mapping of $\widetilde{LC}(D)$ and the zero mapping, which implies that $\widetilde{LC}(D)$ is acyclic.

Next we define a subdivision homomorphism $S_n \colon LC_n(D) \to LC_n(D)$ by induction on n. For $n = 0$ let $S_0 = \mathrm{id}$. Suppose $n > 0$ and S_{n-1} is defined. Suppose $f = (f_0, \ldots, f_n)$ a generator of $LC_n(D)$. Let b be a barycentre of Δ_n, denote $b_f = f(b)$. Define

$$S_n(f) = b_f(S_{n-1}(\partial f))$$

and extent S_n to a unique homomorphism $LC_n(D) \to LC_n(D)$.
Next we build a chain homotopy $H_n \colon LC_n(D) \to LC_{n+1}(D)$ between S and id. If you pay attention, you might notice at this point that we did not prove that S is a chain mapping. However it turns out that the existence of H already implies that S is chain mapping (exercise 3.19).

We define H by induction on n. For $n = 0$ we assert $H_0 = 0$ and for $n > 0$

$$H_n(f) = b_f(f - H_{n-1}\partial f).$$

Next we check by induction if $\partial_{n+1}H_n + H_{n-1}\partial_n = \mathrm{id} - S$. For $n = 0$ this is clear, since $H_0 = H_{-1} = \mathrm{id} - S = 0$. Assume the formula is true for $n > 0$. Then

$$\partial_{n+1}H_n(f) = \partial_{n+1}(b_f(f - H_{n-1}\partial f)) = f - H_{n-1}\partial f - b_f(\partial_n(f - H_{n-1}\partial f)), \text{ since}$$

$$\partial_{n+1}b_f = \mathrm{id} - b_f\partial_n.$$

Notice that this is true, because $n \geq 1$. On the other hand by induction we have

$$\partial_n H_{n-1} = \mathrm{id} - S - H_{n-2}\partial_{n-1},$$

so

$$\partial_n(f - H_{n-1}\partial f) = \partial_n f - (\partial_n f - S\partial f - H_{n-2}\partial_{n-1}\partial_n f) = S\partial f.$$

Also $b_f(S\partial f) = S(f)$ by definition. Hence

$$\partial_{n+1}H_n(f) = f - H_{n-1}\partial_f - b_f(S\partial f) = f - S(f) - H_{n-1}\partial(f),$$

so H is a chain homotopy between id and S.

Next we finally look at the general case. Let X be an arbitrary topological space. Define **barycentric subdivision operator** $S \colon C_n(X) \to C_n(X)$ on

generators by $S\sigma = \sigma_\sharp S(\Delta_n)$. Here $S(\Delta_n)$ is the image of $\mathrm{id}\colon \Delta_n \to \Delta_n$, which is an element of $LC_n(\Delta_n)$, so $S\colon LC_n(\Delta_n) \to LC_n(\Delta_n) \subset C_n(\Delta_n)$ is already defined above.
Notice that as an element of $C_n(X)$ $S\sigma$ is a signed sum of the restrictions of σ on all the different simplices in the first barycentric division of Δ_n.
We also define a homotopy $H_n\colon C_n(X) \to C_{n+1}X$ by

$$H_n(\sigma) = \sigma_\sharp(H_n(\Delta_n)),$$

where again $H_n(\Delta_n)$ is the image of $\mathrm{id}\colon \Delta_n \to \Delta_n$ under already defined $H_n\colon LC_n(\Delta_n) \to LC_{n+1}(\Delta_n) \subset C_n(\Delta_n)$. The corresponding property of this homotopy shows that H is then a chain homotopy between id and S (exercise 3.21), which also shows that S is a chain mapping (exercise 3.20).

Suppose $\mathcal{U}$ is a covering of X such that $\mathrm{int}\,\mathcal{U} = \{\mathrm{int}\,U \mid U \in \mathcal{U}\}$ is also a covering of X.
Let $\sigma \in \mathrm{Sing}_n(X)$ be a singular n-simplex. Since $\sigma^{-1}(\mathrm{int}\,\mathcal{U})$ is an open covering of Δ_n, there exists $m \in \mathbb{N}$ such that the m-th barycentric division of Δ_n is finer than this covering. It follows that iterated barycentric subdivision operator S^m maps σ onto an element of $C_n^{\mathcal{U}}(X)$. Of course m depends on σ. For every $\sigma \in \mathrm{Sing}_n(X)$ we denote by $m(\sigma)$ the smallest integer $\in \mathbb{N}$ with this property. Notice that for every face $\partial_n^i(\sigma)$ we have

$$m(\partial_n^i(\sigma)) \leq m(\sigma).$$

Also for any $m \geq m(\sigma)$ evidently $S^m(\sigma) \in C_n^{\mathcal{U}}(X)$.

Exercise 3.15 implies that

$$D_m = \sum_{0 \leq i < m} HS^i$$

is a chain homotopy from id to S^m for every $m \in \mathbb{N}$.
We define $D\colon C_n(X) \to C_{n+1}(X)$ for every $n \in \mathbb{N}$ by $D(\sigma) = D_{m(\sigma)}(\sigma)$. Now

$$(\partial D + D\partial)(\sigma) = (\partial D_{m(\sigma)}(\sigma) + D_{m(\sigma)}(\partial\sigma)) - D_{m(\sigma)}(\partial(\sigma)) + D(\partial(\sigma)) =$$

$$= (\sigma - S^{m(\sigma)}) - D_{m(\sigma)}(\partial(\sigma)) + D(\partial(\sigma)) = \sigma - p(\sigma),$$

where we used the equation

$$\partial D^{m(\sigma)} + D^{m(\sigma)}\partial = \mathrm{id} - S^{m(\sigma)}$$

and denoted $p(\sigma) = S^{m(\sigma)} + D_{m(\sigma)}(\partial(\sigma)) - D(\partial\sigma)$. We claim that $p(\sigma) \in C_n^{\mathcal{U}}(X)$. This is clear for the term $S^{m(\sigma)}$. For all $i \in \{0, \dots, n\}$ we have $m(\partial^i\sigma) \leq m(\sigma)$, so

$$D_{m(\sigma)}(\partial^i\sigma) - D(\partial^i\sigma) = D_{m(\sigma)}(\partial^i\sigma) - D_{m(\partial^i\sigma)}(\partial_i\sigma) = \sum_{m(\partial^i\sigma)\leq j<m(\sigma)} HS^j(\partial\sigma).$$

Now for $j \geq m(\partial^i\sigma)$ $S^j(\partial^i\sigma) \in C_n^{\mathcal{U}}(X)$, as we already noticed above. Also it is easy to see that homotopy H maps $C_n^{\mathcal{U}}(X)$ into itself (check), so the claim is proved. If we now denote by p the mapping $C(X) \to C^{\mathcal{U}}(X)$ defined by p, we see that D is then a homotopy between id and $i \circ p$, where $i\colon C^{\mathcal{U}}(X) \to C(X)$ is an inclusion.
In a usual way exercise 3.20 implies that $i \circ p$, hence p itself, are chain mappings. Moreover by the definition of p it follows easily that $p \circ i = \mathrm{id}$, because $m(\sigma) = 0 = m(\partial^i\sigma)$ for $\sigma \in C^{\mathcal{U}}(X)$, so $S^{m(\sigma)}(\sigma) = \mathrm{id}(\sigma)$, and $D_{m(\sigma)}(\partial\sigma) = D(\partial\sigma)$.

We have thus prove that p is a chain homotopy inverse of i. This concludes the proof of the theorem 3.3.6, hence also the proof of the excision property.

3.4 The equivalence of the simplicial and singular homologies

In this section we will finally prove the long promised result, that guarantees, that simplicial and singular homologies give the same result for the (finite) Δ-complexes.

First we investigate the structure of $H_m(\Delta_n, \partial\Delta_n)$ for all $m, n \in \mathbb{N}$. Of course we can already compute from the long exact reduced homology sequence of the pair $(\Delta_n, \partial\Delta_n)$ that

$$\partial\colon H_m(\Delta_n, \partial\Delta_n) \to \widetilde{H}_{m-1}(\partial\Delta_n)$$

is an isomorphism, since Δ_n is contractible, so its reduced homology is zero in all dimensions. Also, $\partial\Delta_n$ is homeomorphic to the sphere S^{n-1}, and we calculated the homology of S^{n-1} in the previous section. Hence the theorem 3.3.2 implies that

$$H_m(\Delta_n, \partial\Delta_n) \cong \begin{cases} \mathbb{Z}, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases} .$$

However this is not enough - we need to now which element of $H_n(\Delta_n, \partial\Delta_n)$ is a free generator of this group.
Consider the singular n-simplex $\mathrm{id}\colon \Delta_n \to \Delta_n$, which is an element of $C_n(\Delta_n)$. Then its class in the quotient group $C_n(\Delta_n, \partial\Delta_n)$ is a cycle, since $\partial(\mathrm{id}) \in C_n(\partial\Delta_n)$. Hence there exists its homology class, which we will denote by $[\mathrm{id}] \in H_n(\Delta_n, \partial\Delta_n)$. It turns out that this element is a free generator of $H_n(\Delta_n, \partial\Delta_n)$. This is not surprising, keeping in my mind the result we want to prove, since on the simplicial level $H_n(K(\Delta_n), K(\partial\Delta_n))$ is trivially seen to be a free group generated by the n-simplex Δ_n (or rather its class). Since $i(\Delta_n) = \mathrm{id}$ for the inclusion $i\colon C_n(K(\Delta_n)) \to C_n(\Delta_n)$, id MUST be a generator on the singular homology level, if we want i to induce an isomorphism in homology.

Proposition 3.4.1. [id] *is a generator of the free group* $H_n(\Delta_n, \partial\Delta_n) \cong \mathbb{Z}$.

Proof. We prove the claim by induction on n. For $n = 0$ the pair $(\Delta_n, \partial\Delta_n)$ is just a singleton $(\{0\}, \emptyset) = \{0\}$. Now $H_0(\{0\})$ is a free group generated by a point $[0]$, so we are done.

Suppose $n > 0$. Denote

$$\Lambda_n^0 = \bigcup_{i>0} \partial_n^i \Delta_n,$$

which is thus the union of all $n-1$-faces of Δ_n, except for the 0-th face. The homotopy $\alpha\colon \Delta_n \times I \to \Delta_n$, which contracts Δ_n into the point e_0,

$$\alpha(x, t) = (1 - t)x + te_0$$

has the property $\alpha(\Lambda_n^0 \times I) \subset \Lambda_n^0$ (exercise 3.24). In particular Λ_n^0 is contractible, so its reduced groups are trivial, just as reduced groups of Δ_n. Hence from the long exact reduced homology sequence of the pair (Δ_n, Λ_n^0) it follows that also $H_m(\Delta_n, \Lambda_n^0) = 0$ for all $m \in \mathbb{Z}$. Consider the long exact homology sequence of the triple $(\Delta_n, \partial\Delta_n, \Lambda_n^0)$,

$$H_n(\Delta_n, \Lambda_n^0) \longrightarrow H_n(\Delta_n, \partial\Delta_n) \xrightarrow{\partial} H_{n-1}(\partial\Delta_n, \Lambda_n^0) \xrightarrow{i_*} H_{n-1}(\Delta_n, \Lambda_n^0).$$

It follows that $\partial\colon H_n(\Delta_n, \partial\Delta_n) \to H_{n-1}(\partial\Delta_n, \Lambda_n^0)$ is an isomorphism for all $m \in \mathbb{N}$. By the definition of the boundary operator in the long exact homology sequence it is easy to see that

$$\partial([\mathrm{id}]) = [\partial(\mathrm{id})] = [\sum_{i=0}^{n} \partial^i(id)] = [\partial^0 \,\mathrm{id}] \in H_{n-1}(\partial\Delta_n, \Lambda_n^0),$$

where we also used the obvious fact that $\partial^i \operatorname{id} \in C(\Lambda_n^0)$ for $i > 0$. On the other hand there is a mapping of pairs $\varepsilon^0\colon (\Delta_{n-1}, \partial\Delta_{n-1}) \to (\partial\Delta_n, \Lambda_n^0)$, and clearly $\varepsilon(\operatorname{id}_{n-1}) = \partial^0 \operatorname{id}$. By induction $[\operatorname{id}_{n-1}]$ is a generator of $H_{n-1}(\Delta_n, \partial\Delta_{n-1})$, so to conclude the proof it it enough to prove that $\varepsilon_*\colon H_{n-1}(\Delta_n, \partial\Delta_{n-1}) \to H_{n-1}(\partial\Delta_n, \Lambda_n^0)$ is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} & & H_{n-1}(\partial\Delta_n, \Lambda_n^0) \\ & \overset{\varepsilon_*}{\nearrow} & \uparrow i_* \\ H_{n-1}(\Delta_n, \partial\Delta_{n-1}) & & \\ & \underset{\varepsilon_*}{\searrow} & \\ & & H_{n-1}(\partial\Delta_n \setminus \{e_0\}, \Lambda_n^0 \setminus \{e_0\}) \end{array},$$

where we denote by ε also the mapping $(\Delta_{n-1}, \partial\Delta_{n-1}) \to (\partial\Delta_n \setminus \{e_0\}, \Lambda_n^0 \setminus \{e_0\})$ with restricted image. By choosing $A = \{e_0\}$, $U = \Lambda_n^0$ we see that $A = \overline{A} \subset \operatorname{int} U = \{x \in \partial\Delta_n | x_0 > 0\}$, so $i\colon (\partial\Delta_n \setminus \{e_0\}, \Lambda_n^0 \setminus \{e_0\}) \to (\partial\Delta_n, \Lambda_n^0)$ is an excision mapping, in particular $i_*\colon H_{n-1}(\partial\Delta_n \setminus \{e_0\}, \Lambda_n^0 \setminus \{e_0\}) \to H_{n-1}(\partial\Delta_n, \Lambda_n^0)$ is an isomorphism. Hence it remains to show that $\varepsilon_*\colon H_{n-1}(\Delta_{n-1}, \partial\Delta_{n-1}) \to H_{n-1}(\partial\Delta_n \setminus \{e_0\}, \Lambda_n^0 \setminus \{e_0\})$ is an isomorphism. Define $\lambda\colon (\partial\Delta_n \setminus \{e_0\}, \Lambda_n^0 \setminus \{e_0\}) \to (\Delta_{n-1}, \partial\Delta_{n-1})$ by

$$\lambda(x_0, \dots, x_n) = \Big(\frac{x_1}{1 - x_0}, \dots, \frac{x_n}{1 - x_0}\Big).$$

We leave it to the reader to prove that λ is well-defined and is a homotopy inverse of ε.
Since ε is a homotopy equivalence, it follows that ε_* is an isomorphism. □

Lemma 3.4.2. *Suppose K is a finite Δ-complex and $n \in \mathbb{N}$. Then the inclusion mapping $i\colon C(K^n, K^{n-1}) \to C(|K^n|, |K^{n-1}|)$ induces isomorphism*

$$i_*\colon H_m(K^n, K^{n-1}) \to H_m(|K^n|, |K^{n-1}|)$$

in homology for every $m \in \mathbb{N}$

Proof. The homology groups $H_m(K^n, K^{n-1})$ are calculated in the exercise 2.16. The result is that $H_m(K^n, K^{n-1}) = 0$ for $m \neq n$ and $H_n(K^n, K^{n-1})$ is a free abelian group generated by the set

$$\{[\sigma] \mid \sigma \in K_n/\sim\}.$$

Since $i(\sigma) = f_\sigma$ (the characteristic mapping of σ in $|K|$), the lemma is proved if we can show that $H_m(|K^n|, |K^{n-1}|) = 0$ for $m \neq n$ and $H_n(|K^n|, |K^{n-1}|)$

is a free abelian group with the set of generators $\{[f_\sigma] \mid \sigma \in K_n/\sim]\}$.

For every geometrical n-simplex σ of $|K|$ choose a point $x_\sigma \in \operatorname{int}\sigma$ (for instance one can choose a barycentre). Denote $A = |K^{n-1}|$, $U = |K| \setminus \{x_\sigma \mid \sigma \in K_n\}$. Then A is closed, U is open and $A \subset U$. The inclusion $j\colon (|K^n|,\ |K^{n-1}|) \to (|K^n|,\ U)$ induces isomorphisms in homology for all dimensions (Exercise 3.13).

By excision the inclusion $(|K^n| \setminus |K^{n-1}|,\ U \setminus |K^{n-1}|) \to (|K^n|, U)$ induces isomorphisms in homology. However $(|K^n| \setminus |K^{n-1}|)$ is a disjoint union of interiors $\operatorname{int}\sigma$, where σ goes through all n-dimensional geometric simplices of K and we know that the restriction of the characteristic mapping $f_\sigma\colon \operatorname{int}\Delta_n \to \operatorname{int}\sigma$ is a homeomorphism for all $\sigma \in K_n/\sim$. Hence

$$\oplus((f_\sigma)_*)\colon \oplus H_m(\operatorname{int}\Delta_n, \operatorname{int}\Delta_n \setminus \{b\}) \to H_m(|K^n|,\ U)$$

is an isomorphism for all $m \in \mathbb{N}$. Now the inclusion $(\operatorname{int}\Delta_n, \operatorname{int}\Delta_n \setminus \{b\}) \to (\Delta_n, \Delta_n \setminus \{b\})$ satisfies the excision condition - choose $A = \partial\Delta_n$ and $U = \Delta_n \setminus \{b\})$, and the diagram

$$\begin{array}{ccc}
H_m(\operatorname{int}\Delta_n, \operatorname{int}\Delta_n \setminus \{b\}) & & \\
 & \searrow^{(f_\sigma)_*} & \\
\downarrow \cong & & H_m(|K^n|, U) \\
 & \nearrow_{(f_\sigma)_*} & \\
H_m(\Delta_n, \Delta_n \setminus \{b\}) & &
\end{array},$$

commutes (since it commutes on the level of spaces). Also inclusion $(\Delta_n, \partial\Delta_n) \to (\Delta_n, \Delta_n \setminus \{b\})$ induces isomorphism in homology (exercise 3.13), so we can substitute $H_m(\Delta_n, \Delta_n \setminus \{b\})$ with $H_m(\Delta_n, \partial\Delta_n)$. Combining all these result shows that

$$k = \oplus((f_\sigma)_*)\colon \oplus H_m(\Delta_n, \partial\Delta_n) \to H_m(|K^n|, |K^{n-1}|)$$

is an isomorphism. Previous lemma then implies that $H_m(|K^n|, |K^{n-1}|) = 0$ for $m \neq n$ and $H_n(|K^n|, |K^{n-1}|)$ is a free abelian group with the set of generators $\{[f_\sigma] \mid \sigma \in K_n/\sim\}$. □

Theorem 3.4.3. *Suppose (K, L) is a pair of Δ-complexes. Then the inclusion $i\colon C(K, L) \to C(|K|, |L|)$ induces isomorphisms in homology.*

Proof. The commutative diagram

$$\begin{array}{ccccccccc}
H_n(L) & \longrightarrow & H_n(K) & \longrightarrow & H_n(K,L) & \xrightarrow{\partial} & H_{n-1}(L) & \longrightarrow & H_{n-1}(K) \\
\downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} & & \downarrow & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*} \\
H_n(|L|) & \longrightarrow & H_n(|K|) & \longrightarrow & H_n(|K|,|L|) & \xrightarrow{\partial} & H_{n-1}(|L|) & \longrightarrow & H_{n-1}(|K|)
\end{array}$$

with exact rows and the five-lemma show that it it enough to prove the absolute case.

We will prove the theorem for finite K. The general case is left to the reader (exercises 3.25, 3.26).
The proof is by induction - we show that the theorem is true for K^n for all $n \in \mathbb{N}$.

For $n = 0$ the claim follows from the previous lemma, since $(K^0, K^{-1}) = (K^0, \emptyset)$.
Now suppose the claim is proved for $n - 1$. The following commutative diagram with exact rows, the previous lemma and the five-lemma imply the claim for n.

$$\begin{array}{ccccccccc}
H_{n+1}(K^n, K^{n+1}) & \xrightarrow{\partial} & H_n(K^{n-1}) & \longrightarrow & H_n(K^n) & \longrightarrow & H_n(K^n, K^{n-1}) & \longrightarrow & H_{n-1}(K^{n-1}) \\
\downarrow{\scriptstyle i_*\cong} & & \downarrow{\scriptstyle i_*\cong} & & \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_*\cong} & & \downarrow{\scriptstyle i_*} \\
H_{n+1}(|K^n|, |K^{n-1}|) & \longrightarrow & H_n(|K^{n-1}|) & \longrightarrow & H_n(|K^n|) & \longrightarrow & H_n(|K^n|, |K^{n-1}|) & \longrightarrow & H_{n-1}(|K^{n-1}|)
\end{array}$$

□

The result is quite powerful indeed - in many cases the calculation of simplicial homology is much simpler and concrete algebra concerning (finitely generated) free abelian groups, and we have seen many examples of this. The result also helps further investigation of the structure of homological groups. As an example let us calculate the concrete generator for $H_n(S^n), n > 0$.

Example 3.4.4. *Of course $S^n \cong \partial\Delta_{n+1}$ and we already know that*

$$[\partial\,\mathrm{id}] = \sum_{i=0}^{n} (-1)^{n+1} [\varepsilon^i_{n+1}]$$

is a generator of $H_n(\partial\Delta_{n+1})$, since $\partial\colon H_{n+1}(\Delta_{n+1}, \partial\Delta_{n+1}) \to H_n(\partial\Delta_{n+1})$ is an isomorphism and $[id]$ is a generator of $H_{n+1}(\Delta_{n+1}, \partial\Delta_{n+1})$. Thus for any homeomorphism $f\colon \partial\Delta_{n+1} \cong S^n$, the element $f_*([\partial\,\mathrm{id}])$ is a generator *of $H_n(S^n)$, but it depends on the choice of the homeomorphism and in many cases is inconvenient to use, since it does not have a simple and clear relation to the structure of S^n.*

Let

$$B_+ = \{x \in S^n | x_{n+1} \geq 0\},$$

$$B_- = \{x \in S^n | x_{n+1} \leq 0\}.$$

Define $\iota\colon S^n \to S^n$, $\iota(x_0, \dots, x_n, x_{n+1}) = (x_0, \dots, x_n, -x_{n+1})$. Then ι is clearly a homeomorphism which takes B_+ to B_- (and vice versa).
Exercise 3.16 shows that $\alpha\colon B_+ \to \overline{B}^n$,

$$\alpha(x) = (x_1, \dots, x_n)$$

is a homeomorphism. Choose any homeomorphism $\beta\colon \Delta_n \to \overline{B}^n$, then $f = \alpha^{-1} \circ \beta\colon \Delta^n \to B_+ \subset S^n$ is a homeomorphism and can be thought of as an element of $C_n(S^n)$. Also $g = \iota \circ f\colon \Delta_n \to B_-$ is a homeomorphism, that can be identified with an element of $C_n(S^n)$.
It is easy to see that the images of f and g intersect precisely at the "equator"

$$S^{n-1} = B_+ \cap B_- = \{x \in S^n \mid x_{n+1} = 0.\}$$

which is the image of the boundaries of Δ_n under both mappings. Hence if we take two n-simplices U and V and identify all their $(n-1)$-faces i.e. $\partial^i U$ is identified with $\partial^i V$, we obtain a Δ-complex K such that $|K| = S^n$. Mappings f and g are then precisely characteristic mappings of U and V.
Now $\partial U = \partial V \neq 0$ in $C_{n-1}(K)$, so $H_n(K) = \operatorname{Ker} \partial_n$ (since there are no $n+1$-simplices) and an element $kU + lV \in C_n(K)$ is in the kernel of ∂_n if and only if

$$\partial(kU + lV) = (k+l)\partial U = 0 \text{ i.e. if and only if } k + l = 0.$$

Hence $H_n(K)$ is a free abelian group generated on one element $U - V$. Using the isomorphism $i_*\colon H_n(K) \to H_n(|K|)$ we see immediately that $[f - g]$ is a generator of $H_n(S^n)$.

Hence geometrically $H_n(S^n)$ is generated by the "upper hemisphere minus lower hemisphere".

An interesting corollary is that for the mapping $\iota_*\colon H_n(S^n) \to H_n(S^n)$ induced by ι we have $\iota_*(x) = -x$ for all $x \in H_n(S^n)$. Indeed it is enough to *prove it for the generator* $[f-g]$*, but* $\iota_\sharp(f) = \iota \circ f = g$ *and* $\iota_\sharp(g) = \iota \circ g = \iota \circ \iota \circ f = f$*, since* $\iota \circ \iota = \mathrm{id}$*. Hence*

$$\iota_*[f-g] = [g-f] = -[f-g].$$

Using this fact it is now easy to prove (exercise 3.28) that for the antipodal mapping $h\colon S^n \to S^n$*,* $h(x) = -x$ *one has*

$$h_*(x) = (-1)^{n+1}x, \ x \in H_n(S^n)$$

3.5 Mayer-Vietoris sequence.

Mayer-Vietoris sequence is another way to formalize properties of the singular homology connected to the excision property. In some contexts it can be very convenient from the technical point of view.

Algebraic motivation behind the Mayer-Vietoris sequence is quite simple. Suppose A and B are both subgroups of an abelian group G. Then $A+B = \{a+b \mid a \in A, b \in B\}$ is also a subgroup of G. In general we cannot expect this sum to be a direct sum $A \oplus B$, since the intersection $A \cap B$ might be non-trivial. Nevertheless there is a natural group homomorphism $j\colon A \oplus B \to A + B$ defined by

$$j(a,b) = a+b.$$

Clearly j is a surjection. An element $(a,b) \in A \oplus B$ is in the kernel of j if and only $a+b = 0$ i.e. $a = -b$, in which case $a, b \in A \cap B$. Hence if we define $i\colon A \cap B \to A \oplus B$ by $i(x) = (x, -x)$, then $\mathrm{Im}\, i = \mathrm{Ker}\, j$. Moreover one easily sees that i is injection. Hence we have a short exact sequence

$$0 \longrightarrow A \cap B \xrightarrow{i} A \oplus B \xrightarrow{j} A + B \longrightarrow 0$$

of abelian groups and homomorphisms.

Suppose (C, ∂) is a chain complex and $A, B \subset C$ are subcomplexes of C. Then $A \cap B$ and $A + B$ are also subcomplexes and, by the considerations above, we have a short exact sequence

$$0 \longrightarrow A_n \cap B_n \xrightarrow{i_n} A_n \oplus B_n \xrightarrow{j_n} A_n + B_n \longrightarrow 0,$$

for every $n \in \mathbb{N}$, where i_n and j_n are defined as above. It is easy to check that $i = \{i_n\}$ and $j = \{j_n\}$ are chain mappings. Hence we have a short exact sequence

$$0 \longrightarrow A \cap B \xrightarrow{i} A \oplus B \xrightarrow{j} A + B \longrightarrow 0$$

of chain complexes and chain mappings. This induces long exact sequence

$$\dots \longrightarrow H_{n+1}(A+B) \xrightarrow{\partial} H_n(A\cap B) \xrightarrow{i_*} H_n(A)\oplus H_n(B) \xrightarrow{j_*} H_n(A+B) \longrightarrow \dots,$$

in homology. Here we used the fact that homology commutes with direct sums (Lemma 3.1.2).
Notice that the boundary operator in the long exact sequence is defined as following. Suppose $c \in A+B$ is a cycle. Then $c = a+b$ for some $a \in A, b \in B$ (not necessarily unique) and $\partial_n(a)+\partial_n(b) = 0$. Then in the long exact homology sequence

$$\partial[c] = [\partial_n a] = -[\partial_n(b)].$$

Not let us apply these constructions to the singular homology. Suppose X is a topological space and $U, V \subset X$. Then $C(U)\cap C(V) = C(U\cap V)$, hence we have the long exact sequence

$$\dots \longrightarrow H_{n+1}(C(U)+C(V)) \xrightarrow{\partial} H_n(U\cap V) \xrightarrow{i_*} H_n(U)\oplus H_n(V)$$

$$H_n(C(U)+C(V)) \xrightarrow{\partial} H_{n-1}(U\cap V) \longrightarrow \dots$$

defined as above.
Now suppose inclusion $i\colon C(U)+C(V)\to C(X)$ is such that it induces isomorphism i_* in homology. In this case we call a triple $(X;U,V)$ **a proper triad**.
For example the theorem 3.3.6 implies that $(X;U,V)$ is a proper triad if $\operatorname{int} U \cup \operatorname{int} V = X$.

We have proved the following result.

Proposition 3.5.1. *Suppose $(X;U,V)$ is a proper triad. Then there is an exact sequence*

$$\dots \longrightarrow H_{n+1}(X) \xrightarrow{\partial} H_n(U\cap V) \xrightarrow{i_*} H_n(U)\oplus H_n(V) \xrightarrow{j_*} H_n(X) \xrightarrow{\partial} H_{n-1}(U\cap V) \longrightarrow \dots,$$

called the **Mayer-Vietoris sequence** *of the triple $(X;U,V)$.*

Mayer-Vietoris sequence is easily seen to be natural with respect to the mappings of proper triads. To be precise let $(X;U,V)$ and $(Y;Z,W)$ be proper triads and suppose $f\colon X\to Y$ is a continuous mapping with $f(U)\subset$

Z, $f(V) \subset W$. We also notate this as $f\colon (X;U,V) \to (Y;Z,W)$. Then the diagram

$$\begin{array}{ccccccccccccc}
\ldots & \longrightarrow & H_{n+1}(X) & \xrightarrow{\partial} & H_n(U\cap V) & \xrightarrow{i_*} & H_n(U)\oplus H_n(V) & \xrightarrow{j_*} & H_n(X) & \xrightarrow{\partial} & H_{n-1}(U\cap V) & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f|_*} & & \downarrow{\scriptstyle f|_*\oplus f|_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f|_*} & & \\
\ldots & \longrightarrow & H_{n+1}(Y) & \xrightarrow{\partial} & H_n(Z\cap W) & \xrightarrow{i_*} & H_n(Z)\oplus H_n(W) & \xrightarrow{j_*} & H_n(Y) & \xrightarrow{\partial} & H_{n-1}(Z\cap W) & \longrightarrow & \ldots
\end{array}$$

commutes. The simple verification of this claim is left to the reader.

Example 3.5.2. *As an example of the way Mayer-Vietoris sequence can be applied, let us calculate a generator for S^1 (again).*
Consider open subsets $U = S^1 \setminus \{e_2\}$ and $V = S^1 \setminus \{-e_2\}$. Then $U \cup V = S^1$, so the portion of Mayer-Vietoris sequence

$$H_1(U)\oplus H_1(V) \xrightarrow{j_*} H_1(S^1) \xrightarrow{\partial} H_0(U\cap V) \xrightarrow{i_*} H_0(U)\oplus H_0(V)$$

is exact. Both U and V are homeomorphic to the open interval $]0,1[$, hence contractible. It follows that $H_1(U) = H_1(V) = 0$, so $H_1(S^1)$ is isomorphic to $\operatorname{Im}\partial = \operatorname{Ker} i_*$ *via the homomorphism ∂. Now $U \cap V$ is not path-connected - it has two path-components - the set $\{x \in S_1 \mid x_1 > 0\}$ and the set $\{x \in S_1 \mid x_1 < 0\}$. It follows that $H_0(U\cap V) = \mathbb{Z}[e_1] \oplus \mathbb{Z}[-e_1]$ is a free abelian group on 2 generators, where we choose as representatives of corresponding path-components points e_1 and $-e_1$. On other hand U and V are both path-connected and $i_*[e_1] = ([e_1],[e_1]) = y$, $i_*[-e_1] = ([-e_1],[-e_1]) = ([e_1],[e_1]) = y$, so*

$$i_*(n[e_1] + m[-e_1]) = (n+m)y.$$

Here we used the fact that both U and V are path-connected so the classes of $[e_1]$ and $[-e_1]$ coincide in both $H_0(U)$ and $H_0(V)$.
It follows that $\operatorname{Ker} i_* = \operatorname{Im}\partial$ *is a free abelian group generated by $[e_1] - [-e_1]$.*
Now to find a generator for $H_1(S^1)$ all we need to do is to find a cycle z such that $\partial[z] = [e_1] - [-e_1]$.
One possibility is indicated in the picture below. Using complex number notation for the points of S^1 we define $\alpha, \beta\colon I \to S^1$ by

$$\alpha(t) = e^{\pi t i} = \cos(\pi t) + i\sin(\pi t),$$

$$\beta(t) = e^{(\pi+\pi t)i} = \cos(\pi + \pi t) + i\sin(\pi+\pi t).$$

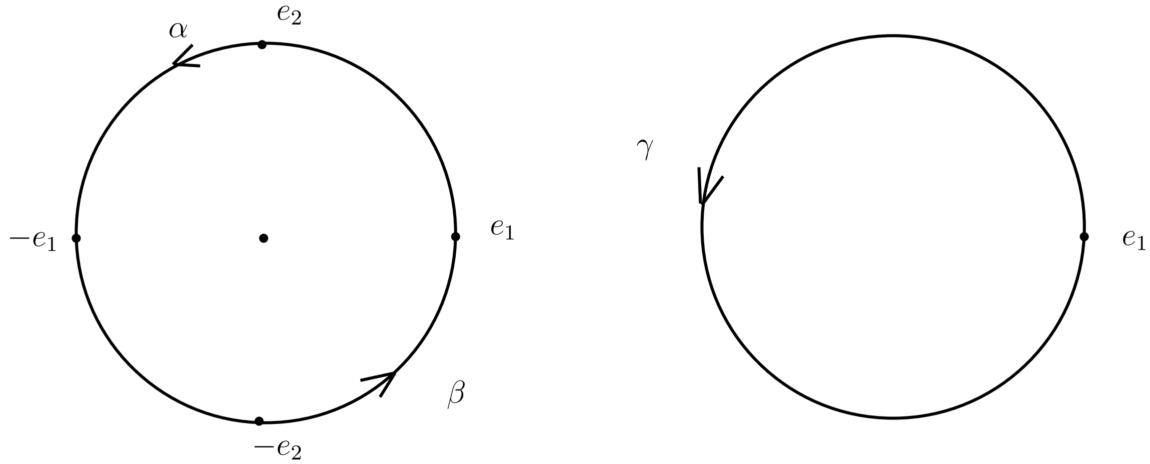
Then $x = \alpha + \beta \in C_1(S^1)$ is a cycle ($\partial(\alpha) = e_1 - (-e_1)$ and $\partial\beta = (-e_1) - e_1$) and $\partial([x]) = [\partial_1(\alpha)] = [e_1] - [-e_1]$, since $\alpha \in C_1(V)$, $\beta \in C_1(U)$.
Hence $[\alpha+\beta]$ is a generator of $H_1(S^1)$. Now if we "compose" pathes α and β, we obtain a path $\gamma = \alpha \cdot \beta$, defined by

$$\gamma(t) = e^{2\pi t i} = \cos(\pi(t)) + i\sin(\pi t),$$

which is just one round around S^1 in the counter-clockwise direction. Exercise 2.11 implies that

$$[\gamma] = [\alpha + \beta],$$

hence $[\gamma]$ is also generator for $H_1(S^1)$ (which seems intuitively very expected).



Reader might have a suspicion that the calculation above would be easier if one would use reduced groups instead of absolute groups. It is true - there is also a reduced version of the Mayer-Vietoris sequence

$$\ldots \longrightarrow \tilde{H}_{n+1}(X) \xrightarrow{\partial} \tilde{H}_n(U \cap V) \xrightarrow{i_*} \tilde{H}_n(U) \oplus \tilde{H}_n(V) \xrightarrow{j_*} \tilde{H}_n(X) \xrightarrow{\partial} \tilde{H}_{n-1}(U \cap V) \longrightarrow \ldots$$

which is naturally called **the reduced Mayer-Vietoris exact sequence of the proper triad** $(X; U, V)$. Of course one has to assume that $U \cap V = \emptyset$, so that all reduced groups are defined. The proof of existence of the reduced Mayer-Vietoris exact sequence is left to the reader (exercise 3.27).

Example 3.5.3. *As a more complicated example let us calculate the homology of the projective plane $\mathbb{R}P^n$ for all $n \in \mathbb{N}$. The traditional way to define projective plane is to say that $\mathbb{R}P^n$ is a quotient space of S^n defined by the equivalence relation $\sim$, which is generated by relations $x \sim -x$ for all $x \in S^n$. There is also another model for $\mathbb{R}P^n$. Define an equivalence relation $\sim'$ on the closed ball $\overline{B}^n$ generated by the relations $x \sim' -x$ for all $x \in S^{n-1}$. Notice that the identifications happen only on the boundary and the open ball B^n remain "untouched". In other words if $q\colon \overline{B}^n \to \overline{B}^n/\sim' = X$ is a quotient mapping, then its restriction to B^n is a homeomorphism to its image, which we will denote as U. U is clearly open in X. We also denote $V = q(\overline{B}^n \setminus \{0\})$, V is open in X as well. Clearly $U \cup V = X$. Hence there is a reduced Mayer-Vietoris exact sequence*

$$\ldots \longrightarrow \widetilde{H}_{m+1}(X) \xrightarrow{\partial} \widetilde{H}_m(U \cap V) \xrightarrow{i_*} \widetilde{H}_m(U) \oplus \widetilde{H}_m(V) \xrightarrow{j_*} \widetilde{H}_m(X) \xrightarrow{\partial} \ldots$$

Define the mapping $p\colon S^n \to \overline{B}^n / \sim'$ *by*

$$p(x_1, \dots, x_{n+1}) = [(x_1, \dots, x_n)] \textit{ if } x_{n+1} \geq 0,$$

$$p(x_1, \dots, x_{n+1}) = [(-x_1, \dots, -x_n)] \textit{ if } x_{n+1} \leq 0.$$

Then p *is well-defined, continuous and induces a homeomorphism* $\mathbb{R}P^n = S^n / \sim \to \overline{B}^n / \sim' = X$. *The proof of this claims is left to the reader as an exercise 3.30. Hence we can identify* $\mathbb{R}P^n = X$. *Also notice that under this identification* $p\colon S^n \to \mathbb{R}P^n$ *is exactly a projection quotient space.*

It follows that the subspace $p(S^{n-1}) \subset \mathbb{R}P^n$ *is homeomorphic to* $\mathbb{R}P^{n-1}$. *Thus we will consider* $\mathbb{R}P^{n-1}$ *as a subspace of* $\mathbb{R}P^n$. *Notice that the inclusion* $\mathbb{R}P^{n-1} \to V$ *is a homotopy equivalence - exactly for the same reason that* $S^{n-1} \hookrightarrow \overline{B}^n \setminus \{0\}$ *is a homotopy equivalence (exercise).*
Hence we can substitute $\widetilde{H}_m(V)$ *above with* $\widetilde{H}_m(\mathbb{R}P^{n-1})$. *Also* U *is contractible (since it is essentially* B^n*), so its reduced groups are trivial. What about* $U \cap V$*? Since* U *is homeomorphic to* B^n, $U \cap V$ *is homeomorphic to the punctured open ball* $B^n \setminus \{0\}$ *in a natural way. Clearly it has the same homotopy type as its subspace*

$$\{x \in B^n \mid |x| = 1/2\},$$

which is homeomorphic to S^{n-1}. *Under these substitutions the inclusion* $U \cap V \to V$ *becomes a quotient projection* $p\colon S^{n-1} \to \mathbb{R}P^{n-1}$. *Hence we obtain an exact sequence*

$$\dots \longrightarrow \widetilde{H}_{m+1}(\mathbb{R}P^n) \xrightarrow{\partial} \widetilde{H}_m(S^{n-1}) \xrightarrow{p_*} \widetilde{H}_m(\mathbb{R}P^{n-1}) \xrightarrow{i_*} \widetilde{H}_m(\mathbb{R}P^n) \xrightarrow{\partial} \widetilde{H}_{m-1}(S^{n-1}) \longrightarrow \dots.$$

Since $\widetilde{H}_{n-1}(S^{n-1})$ *is the only non-trivial reduced homology group of* S^{n-1} *we see that*
1) $i_*\colon \widetilde{H}_m(\mathbb{R}P^{n-1}) \to \widetilde{H}_m(\mathbb{R}P^n)$ *induced by inclusion is an isomorphism for* $m \neq n, n-1$.
2) There is an exact sequence

$$0 \longrightarrow \widetilde{H}_n(\mathbb{R}P^{n-1}) \xrightarrow{i_*} \widetilde{H}_n(\mathbb{R}P^n) \xrightarrow{\partial} \widetilde{H}_{n-1}(S^{n-1}) \xrightarrow{p_*} \widetilde{H}_{n-1}(\mathbb{R}P^{n-1}) \xrightarrow{i_*} \widetilde{H}_{n-1}(\mathbb{R}P^n) \longrightarrow 0.$$

So, if we want to proceed by the induction, we need not only to know the homology groups of $\mathbb{R}P^{n-1}$ *but also a homomorphism*

$$p_*\colon \widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(\mathbb{R}P^{n-1})$$

.

Let us start with $n = 1$. The second model for $\mathbb{R}P^1$ shows immediately that $\mathbb{R}P^1 = [-1,1]/\{1,-1\}$ is a closed interval with end points identified, hence essentially S^1. Moreover $p\colon S^1 \to S^1 = \mathbb{R}P^1$ looks like the mapping that wraps the upper hemisphere of S^1 around S^1 one time, and the same for the lower hemisphere.
Hence if we use α, β, γ from the previous example, we see that $p_\sharp(\alpha) = p_\sharp(\beta) = \gamma$, so

$$p_*([\gamma]) = p_*([\alpha+\beta]) = [\gamma+\gamma] = 2[\gamma].$$

Hence p_* looks like the homomorphism $\mathbb{Z} \to \mathbb{Z}$, $n \mapsto 2n$.

In particular $\widetilde{H}_m(\mathbb{R}P^1) = 0$ for $m > 1$. Since $\widetilde{H}_m(\mathbb{R}P^n)$ is isomorphic to $\widetilde{H}_m(\mathbb{R}P^{n-1})$ for $m > n$, we see immediately by induction that $\widetilde{H}_m(\mathbb{R}P^n) = 0$ for $m > n$. This implies that the exact sequence above becomes the exact sequence

$$0 \longrightarrow \widetilde{H}_n(\mathbb{R}P^n) \xrightarrow{\partial} \widetilde{H}_{n-1}(S^{n-1}) \xrightarrow{p_*} \widetilde{H}_{n-1}(\mathbb{R}P^{n-1}) \xrightarrow{i_*} \widetilde{H}_{n-1}(\mathbb{R}P^n) \longrightarrow 0.$$

Hence $\widetilde{H}_n(\mathbb{R}P^n)$ is a subgroup $\operatorname{Ker} p_*$ of $\widetilde{H}_{n-1}(S^1) \cong \mathbb{Z}$, so in particular it is either trivial, or isomorphic to $\mathbb{Z}$. Hence $p_*\colon \widetilde{H}_{n-1}(S^{n-1}) \to \widetilde{H}_{n-1}(\mathbb{R}P^{n-1})$ is always a homomorphism $\mathbb{Z} \to \mathbb{Z}$, which is either zero or injective. In case it is injective, we have that $\widetilde{H}_n(\mathbb{R}P^n) = \operatorname{Ker} p_* = 0$. In case it is a zero homomorphism, $\widetilde{H}_n(\mathbb{R}P^n) = \operatorname{Ker} p_* = \widetilde{H}_{n-1}(S^{n-1}) \cong \mathbb{Z}$. Hence in particular if $\widetilde{H}_n(\mathbb{R}P^n) = 0$, then in the next dimension we must have $\widetilde{H}_{n+1}(\mathbb{R}P^{n+1}) \cong \mathbb{Z}$ and also $\widetilde{H}_n(\mathbb{R}P^{n+1}) = 0$

Let us continue by induction and consider the next case $n = 2$. Since we have seen that $p_*\colon \widetilde{H}_1(S^1) \to \widetilde{H}_1(\mathbb{R}P^1)$ is an injection, it follows that $\widetilde{H}_2(\mathbb{R}P^2) = \operatorname{Ker} p_* = 0$ and $\widetilde{H}_1(\mathbb{R}P^2) \cong \widetilde{H}_1(\mathbb{R}P^1)/\operatorname{Im} p_* = \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$.
From the considerations from the preceding paragraph it now follows that $\widetilde{H}_3(\mathbb{R}P^3) \cong \mathbb{Z}$, $\widetilde{H}_2(\mathbb{R}P^3) = 0$ and $\widetilde{H}_1(\mathbb{R}P^3) = \widetilde{H}_1(\mathbb{R}P^2) = \mathbb{Z}_2$.
For the next case $n = 4$ we need to know $p_*\colon \mathbb{Z} = \widetilde{H}_3(S^3) \to \widetilde{H}_3(\mathbb{R}P^3) = \mathbb{Z}$, so that us investigate this matter in general.

To make use of the naturality of Mayer-Vietoris sequence it makes sense to define proper triad $(S^n; Z, W)$ such that $p\colon S^n \to \mathbb{R}P^n$ is a mapping of triads. Miming the definition of U and V and keeping in mind the definition of p we define

$$Z = Z_+ \cup Z -_= \{x \in S^n \mid x_{n+1} > 0\} \cup \{x \in S^n \mid x_{n+1} < 0\},$$

$$W = S^n \setminus \{e_{n+1}, -e_{n+1}\}.$$

Then $(S^n;Z,W)$ *is a proper triad and* $p\colon (S^n;Z,W)\to(\mathbb{R}P_n,U,V)$*, so we have a commutative diagram*

$$\begin{array}{ccccccccc}
\widetilde{H}_{m+1}(S^n) & \xrightarrow{\partial} & \widetilde{H}_m(Z\cap W) & \xrightarrow{i_*} & \widetilde{H}_m(Z)\oplus\widetilde{H}_m(W) & \xrightarrow{j_*} & \widetilde{H}_m(S^n)p_* & \xrightarrow{\partial} & \widetilde{H}_{m-1}(Z\cap W) \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p|_*} & & \downarrow{\scriptstyle p|_*\oplus p|_*} & & \downarrow & & \downarrow{\scriptstyle p|_*} \\
\widetilde{H}_{m+1}(\mathbb{R}P^n) & \xrightarrow{\partial} & \widetilde{H}_m(U\cap V) & \xrightarrow{i_*} & \widetilde{H}_m(U)\oplus\widetilde{H}_m(V) & \xrightarrow{j_*} & \widetilde{H}_m(\mathbb{R}P^n) & \xrightarrow{\partial} & \widetilde{H}_{m-1}(U\cap V)
\end{array}$$

with exact rows. Now let us simplify the upper row in the same way we have already simplified the lower row. The subspace $Z\cap W$ *has the same homotopy type as its subspace* $S_+\cup S_-$*, where*

$$S_+=\{x\in S^n \mid x_{n+1}=\sqrt{3}/4\},$$
$$S_-=\{x\in S^n \mid x_{n+1}=-\sqrt{3}/4\},$$

via the inclusion $S_+\cup S_-\to Z\cap W$*. The reason we have chosen the weird looking number* $\sqrt{3}/4$ *above is that then* p *maps* $S_+\cup S_-$ *onto* $\{x\in B^n \mid |x|=1/2\}\subset\mathbb{R}P_n$*. Since both* S_+ *and* S_- *are homeomorphic to* S^{n-1} *in an obvious way, we can write the restriction of* p *to* $S_+\cup S_-$ *as a mapping* $S^{n-1}\sqcup S^{n-1}\to S^{n-1}$ *(*$\sqcup$ *denotes the disjoint topological union) defined by identity mapping* id *on the first copy of* S^{n-1} *(corresponding to* S_+*) and by the antipodal mapping* $h\colon S^{n-1}\to S^{n-1}$*,* $h(x)=-x$ *on the second summond corresponding to* S^{n-1}*. In particular* $p|_*\colon\widetilde{H}_m(Z\cap W)\to\widetilde{H}_m(U\cap V)$ *in the diagram above becomes then* $\mathrm{id}\oplus h_*\colon\widetilde{H}_m(S^{n-1})\oplus\widetilde{H}(S^{n-1})\to\widetilde{H}_m(S^{n-1})$*.*
The subspace Z *is a disjoint union of two contractible spaces, so* $\widetilde{H}_m(Z)=0$ *for* $m>0$*. The subspace* W *has the same homotopy type as its subspace* $S^{n-1}=\{x\in S^n \mid x_{n+1}=0\}$ *and again we see that* $p\colon W\to V$ *becomes* $p\colon S^{n-1}\to\mathbb{R}P^{n-1}$ *as well as* $i\colon Z\cap W\to W$ *becomes* $\mathrm{id}\sqcup\mathrm{id}\colon S^{n-1}\sqcup S^{n-1}\to S^{n-1}$*. Summarizing all these information we obtain the following commutative diagram (for* $n>1$*)*

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H_n(S^n) & \longrightarrow & H_{n-1}(S^{n-1})\oplus H_{n-1}(S^{n-1}) & \xrightarrow{\mathrm{id}\oplus\mathrm{id}} & H_{n-1}(S^{n-1}) & \xrightarrow{j_*} & H_{n-1}(S^n) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle \mathrm{id}\oplus h_*} & & \downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle p_*} & & \\
0 & \longrightarrow & H_n(\mathbb{R}P^n) & \xrightarrow{\partial} & H_{n-1}(S^{n-1}) & \xrightarrow{i_*} & H_{n-1}(RP^{n-1}) & \xrightarrow{j_*} & H_{n-1}(\mathbb{R}P^n) & \xrightarrow{\partial} & 0
\end{array}$$

with exact rows. Now the interesting case is the case when $p_*\colon H_{n-1}(S^{n-1})\to H_{n-1}(\mathbb{R}P^{n-1})$ *is zero mapping (since in the other case it is injective and* $H_n(\mathbb{R}P^n)=0$*). In this case we have the commutative diagram*

$$\begin{array}{ccc}
H_n(S^n) & \xrightarrow{\partial\cong} & \mathrm{Ker}(\mathrm{id}\oplus\mathrm{id}\colon H_{n-1}(S^{n-1})\oplus H_{n-1}(S^{n-1})\to H_{n-1}(S^{n-1})) \\
\downarrow{\scriptstyle p_*} & & \downarrow{\scriptstyle (id_*\oplus h_*)|} \\
H_n(\mathbb{R}P^n) & \xrightarrow{\partial\cong} & H_{n-1}(S^{n-1}).
\end{array}$$

Now $\mathrm{id} \oplus \mathrm{id}\colon H_{n-1}(S^{n-1}) \oplus H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$ *looks like the mapping* $\mathbb{Z} \oplus \mathbb{Z} \to \mathbb{Z}, (n, m) \mapsto n + m$, *so*

$$\mathrm{Ker}(\mathrm{id} \oplus \mathrm{id}) = \{(x, -x) \in H_{n-1}(S^{n-1}) \oplus H_{n-1}(S^{n-1})) \mid x \in H_{n-1}(S^{n-1})\}.$$

According to the exercise 3.27 $h_*(x) = (-1)^n x)$ *for all* $x \in H_{n-1}(S^{n-1})$. *Hence*

$$(\mathrm{id}_* \oplus h_*)(x, -x) = x + (-1)^n(-x) = x + (-1)^{n+1}(x) = \begin{cases} 0, & \text{if } n \text{ is even,} \\ 2x, & \text{if } n \text{ is odd.} \end{cases}$$

Hence the mapping $p_*\colon H_n(S^n) \to H_n(\mathbb{R}P^n)$ *has the same description i.e. as a mapping* $\mathbb{Z} \to \mathbb{Z}$ *looks like the zero mapping if* n *is even and the mapping* $n \mapsto 2n$ *if* n *is odd, at least with the suitable choice of the generators.*
Let us prove by induction that in fact only the second case occurs (when $H_n(\mathbb{R}P^n) \cong \mathbb{Z}$*). To be precise we claim that*
1) if n *is even* $H_n(\mathbb{R}P^n) = 0$ *and* $H_{n-1}(\mathbb{R}P^n) \cong \mathbb{Z}_2$,
2) if n *is odd* $H_n(\mathbb{R}P^n) \cong \mathbb{Z}, \tilde{H}_{n-1}(\mathbb{R}P^n) = 0$ *and* $p_*\colon H_n(S^n) \to H_n(\mathbb{R}P^n)$ *is essentially a mapping* $n \mapsto 2n$.

For $n = 1$ *we have already shown the claim to be true. Suppose* n *is odd and the claim is true for* $n - 1$, *which is then even. Then* $H_{n-1}(\mathbb{R}P^{n-1}) = 0$ *and the considerations above apply, showing that the claim is true also for* n. *If* n *is even and the claim is true for* $n-1$, *which is then odd, then* $p_*\colon H_n(S^{n-1}) \to H_n(\mathbb{R}P^{n-1})$*is the injection with image* $2\mathbb{Z} \subset \mathbb{Z}$, *so the exact sequence above shows that* $H_n(\mathbb{R}P^n) = 0$ *and* $H_{n-1}(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$.

Now we can finally gather all the information about the homology groups of $\mathbb{R}P^n$.

$$H_m(\mathbb{R}P_n) = \begin{cases} \mathbb{Z}, & \text{for } m = 0, \\ \mathbb{Z}_2, & \text{for } 0 < m < n \text{ if } m \text{ is odd }, \\ \mathbb{Z}, & \text{for } m = n \text{ if } n \text{ is odd }, \\ 0, & \text{otherwise.} \end{cases}$$

Before ending this section we notice that the equivalence of the simplicial and singular homologies easily implies the following convenient result.

Proposition 3.5.4. *Suppose* K *is a simplicial complex and* L_1 *and* L_2 *are subcomplexes of* K *such that* $K = L_1 \cup L_2$. *Then* $(|K|; |L_1|, |L_2|)$ *is a proper triad.*

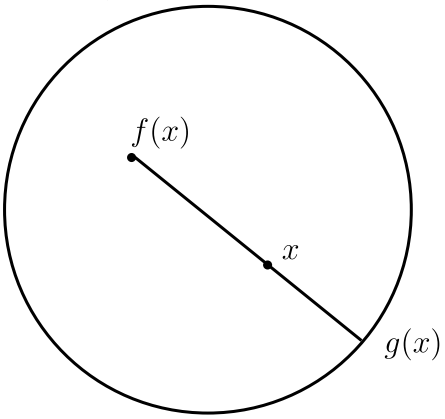
Proof. Exercise 3.31. □

3.6 Some classical applications

In this section we will apply the machinery of singular homology theory to prove some important classical topological results, such as the Brouwer's fixed point theorem, invariance of domain and Jordan-Brouwer separation theorem.

Theorem 3.6.1. The Brouwer's fixed point theorem
Suppose $C \subset V$ is a compact and convex non-empty subset of the finite-dimensional vector space V. Then every continuous mapping $f\colon C \to C$ has a fixed point.

Proof. Since C is homeomorphic to $\overline{B}^n$ for some $n \in \mathbb{N}$, it is enough to consider the case $C = \overline{B}^n$. Let us make a counter-assumption - suppose $f\colon \overline{B}^n \to \overline{B}^n$ is a continuous mapping with a fixed point. Then $f(x) \neq x$ for all $x \in \overline{B}^n$. Consider a half-line L_x starting at $f(x)$ and going through x (see the picture).



For every $x \in \overline{B}^n$ let $g(x)$ be a unique point of S^{n-1} that belongs to L_x. This defines a mapping $g\colon \overline{B}^n \to S^{n-1}$, which is well-defined and continuous (exercise 3.34). By definition it follows that $g(x) = x$ for all $x \in S^{n-1}$. Hence g is a retraction of $\overline{B}^n$ onto S^{n-1}. However this is a contradiction with the corollary 3.3.5, which says that no such retraction can exist. □

There is a generalization of the Brouwer's fixed point theorem, that says that any mapping $f\colon C \to C$, where C is a contractible compact polyhedra, has a fixed point, but the proof is much more difficult and requires the development of some further machinery.

To prove invariance of domain and separation theorem we need some technical results first.

Lemma 3.6.2. *Suppose $B \subset S^n$ is homeomorphic to $\overline{B}^k$ for some $0 \le k \le n$. Then $\tilde{H}_m(S^n \setminus B) = 0$ for all $m \in \mathbb{N}$.*

Proof. The proof is by induction on k. If $k = 0$ B is a point and the claim is clear, since then $S^n \setminus B$ is homeomorphic to $\mathbb{R}^n$, hence contractible.
Suppose $k > 0$ and the claim is true for $k-1$. Since I^k is homeomorphic to $\overline{B}^k$, there is a homeomorphism $f\colon I^k \to B$. Define $C_1 = f(I^{k-1} \times [0, 1/2])$, $C_2 = f(I^{k-1} \times [1/2, 1])$. Since C_1 and C_2 are both compact, $\{S^n \setminus C_1, S^n \setminus C_2\}$ is an open covering of $(S^n \setminus C_1) \cup (S^n \setminus C_2) = S^n \setminus (C_1 \cap C_2) = S^n \setminus C$, where $C = C_1 \cap C_2 = f(I^{k-1} \times \{1/2\})$ is homeomorphic to I^{k-1}. Also $(S^n \setminus C_1) \cap (S^n \setminus C_2) = S^n \setminus B$. From the reduced Mayer-Vietoris sequence

$$\tilde{H}_{m+1}(S^n \setminus C) = 0 \xrightarrow{\partial} \tilde{H}_m(S^n \setminus B) \xrightarrow{(i_*, -i_*)} \tilde{H}_m(S^n \setminus C_1) \oplus \tilde{H}_m(S^n \setminus C_2) \xrightarrow{j_*} \tilde{H}_n(S^n \setminus C) = 0$$

of the proper triad $(S^n \setminus C; S^n \setminus C_1, S^n \setminus C_2)$ and the induction assumption we see that

$$(i_*, -i_*)\colon \widetilde{H}_m(S^n \setminus B) \to \widetilde{H}_m(S^n \setminus C_1) \oplus \tilde{H}_m(S^n \setminus C_2)$$

is an isomorphism.
Fix $m \in \mathbb{N}$. Let us make a counter-assumption, that there is $x \in \widetilde{H}_m(S^n \setminus B)$ such that $x \ne 0$. Then either $i_*(x) \ne 0 \in \widetilde{H}_m(S^n \setminus C_1)$ or $i_*(x) \ne 0 \in \widetilde{H}_m(S^n \setminus C_2)$. Choose $B_1 = C_1$ in the first case and $B_1 = C_2$ in the second case.
Since B_1 satisfies the same assumptions as B (as well as the counter-assumption $\widetilde{H}_m(S^n \setminus B_1) \ne 0$), we may repeat the same reasoning applied to B_1, to obtain B_2. Continuing by induction we obtain a nested sequence

$$B = B_0 \supset B_1 \supset B_2 \supset \ldots \supset B_l \supset B_{l+1} \supset \ldots$$

such that $i^l_*(x) \ne 0$ for all inclusions $i^l\colon (S^n \setminus B) \to (S^n \setminus B_l)$ and $\bigcap_{i=0}^{\infty} B_i = B_\infty$ is homeomorphic to I^{k-1}, hence satisfies the inductive assumption. In other words $\widetilde{H}_m(S^n \setminus B_\infty) = 0$. Let $i\colon (S^n \setminus B) \to (S^n \setminus B_\infty)$ be the inclusion, then $i_*(x) = 0$. Since singular homology theory has compact carriers (exercise 3.25a)), there is compact $C \subset S^n \setminus B$ such that $x = j_*(y)$ for some $y \in \tilde{H}_(C)$, where $j\colon C \to S^n \setminus B$ is an inclusion. It follows that $j'_*(y) = 0 \in \widetilde{H}_m(S^n \setminus B_\infty) = 0$ where $j'\colon C \to S^n \setminus B_\infty$ is an inclusion. By the compact carrier property (exercise 3.25b)) there is compact $D \subset S^n \setminus B_\infty$ such that $C \subset D$ and $j''_*(y) = 0$ for inclusion $j''\colon C \to D$.
Now

$$\{S^n \setminus B_l \mid l \in \mathbb{N}\}$$

is an open covering of $S^n \setminus B_\infty$, hence also of D. Since D is compact there is $q \in \mathbb{N}$ such that

$$D \subset \bigcup_{i=0}^{q} S^n \setminus B_l = S^n \setminus B^q.$$

Since the diagram

$$\begin{array}{ccc} \widetilde{H}_m(C) & \xrightarrow{j''} & \widetilde{H}_m(D) \\ \downarrow j & & \downarrow k_* \\ \widetilde{H}_m(S^n \setminus B) & \xrightarrow{i^q_*} & \widetilde{H}_m(S^n \setminus B^q) \end{array}$$

is commutative ($k\colon D \to S^n \setminus B^q$ inclusion) we see that $i^q_*(x) = 0$. This however contradicts the construction of B^q. Hence counter-assumption was false, so the claim is true also for k. $\square$

Lemma 3.6.3. *Suppose $B \subset S^n$ is homeomorphic to S^k for $0 \leq k \leq n-1$. Then*

$$\widetilde{H}_m(S^n \setminus B) = \begin{cases} 0, & \text{for } m \neq n-k-1, \\ \mathbb{Z}, & \text{for } m = n-k-1. \end{cases}$$

Proof. By induction on k. If $k = 0$, then $B = \{a, b\}$ is two points space and $S^n \setminus B$ is homeomorphic to $\mathbb{R}^n \setminus \{0\}$, so the claim follows.
Suppose claim is true for $k-1 \leq n-2$. Let $f\colon S^k \to B$ be a homeomorpism. Denote $C_1 = f(B_+)$ and $C_2 = f(B_-)$, where B_+, B_- are upper and lower hemisphere in S^k as usual. Then $\{S^n \setminus C_1, S^n \setminus C_2\}$ is an open covering of $S^n \setminus (C_1 \cap C_2) = S^n \setminus C$, where C is homeomorphic to S^{k-1}. Also

$$(S^n \setminus C_1) \cap (S^n \setminus C_2) = S^n \setminus B.$$

By the previous lemma both spaces $S^n \setminus C_1$ and $S^n \setminus C_2$ have trivial reduced groups in all dimensions. The claim now follows from the Mayer-Vietoris sequence

$$\widetilde{H}_m(S^n \setminus C_1) \oplus \tilde{H}_m(S^n \setminus C_2) = 0 \longrightarrow \widetilde{H}_{m+1}(S^n \setminus C) \xrightarrow{\partial} \widetilde{H}_m(S^n \setminus B) \xrightarrow{(i_*, -i_*)} \tilde{H}_m(S^n \setminus C_1) \oplus \tilde{H}_m(S^n \setminus C_2)$$

and induction, since ∂ is isomorphism. $\square$

Theorem 3.6.4. (Jordan-Brouwer separation theorem).
Suppose B is a subset of S^n homeomorphic to S^{n-1}. Then $S^n \setminus B$ has exactly two path components U and V and $B = \partial U = \partial V$.

Proof. The case $k = n-1$ in the previous lemma shows that $\widetilde{H}_0(S^n \setminus B) = \mathbb{Z}$, hence $H_0(S^n \setminus B) = \mathbb{Z} \oplus \mathbb{Z}$, so by Lemma 3.1.5 $S^n \setminus B$ has exactly two path components, U and V. Since $S^n \setminus B$ is open in locally path-connected space S^n, it follows that both U and V are open in S^n. This implies that $\overline{U} \subset S^n \setminus V = U \cup B$, so $\partial U \subset B$. By the same reasoning $\partial V \subset B$.
It remains to prove $B \subset \partial U \cap \partial V$. It is enough to show that $B \subset \overline{U} \cap \overline{V}$. Suppose $x \in B$ and let W be a neighbourhood of $x \in S^n$. Then $B \cap W$ is a neighbourhood of $x \in B$. Since B is homeomorphic to S^{n-1}, there is a neighbourhood A of x in B such that $A \subset B \cap W$ and $B \setminus A$ is homeomorphic to $\overline{B}^{n-1}$.
By Lemma 3.6.2 the set $S^n \setminus (B \setminus A) = U \cup V \cup A$ is path-connected. Hence there exist a path $p\colon I \to S^n \setminus (B \setminus A)$ from $p(0) \in U$ and $p(1) \in V$. Since $U \cup V$ is not path-connected, path p must intersect A. In fact let

$$t_0 = \sup\{t \in I \mid p(t) \in U\}.$$

Then clearly $p(t_0) \in \overline{U}$ and $p(t_0) \notin U \cup V$, so $p(t_0) \in A \cap \overline{U}$. Similarly we see that $A \cap \overline{V} \neq \emptyset$. In particular W intersects both $\overline{U}$ and $\overline{V}$, hence, since W is open, W must intersect both U and V. Since W is an arbitrary neighbourhood of $x \in B$, it follows that $x \in \overline{U} \cap \overline{V}$. The theorem is proved. □

If $n \geq 2$ the Jordan-Brouwer theorem separation also holds in $\mathbb{R}^n$ (what about the case $n = 1$?) i.e. if a subset S of $\mathbb{R}^n$ is homeomorphic to S^{n-1}, then $\mathbb{R}^n \setminus S$ has exactly two components and S is a boundary of both of them. This easily follows from the proved version of the theorem for S^n, since $\mathbb{R}^n$ is homeomorphic to S^n minus a point. Details are left to the reader (exercise 3.35).

Theorem 3.6.5. Invariance of Domain *Suppose U, V are homeomorphic subsets of S^n. If U is open in S^n, also V is.*

Proof. Let $h\colon U \to V$ be a homeomorphism. It is enough to prove that for every $x \in U$ there is a neighbourhood $Z \subset U$ of x such that $h(Z)$ is open in S^n. Let W be a small enough closed ball neighbourhood of x contained in U and let A be its boundary. Then W and $h(W)$ are homeomorphic to $\overline{B}^n$, while A and $h(A)$ are homeomorphic to S^{n-1}. Now

$$S^n = S^n \setminus h(W) \cup h(A) \cup h(W \setminus A)$$

and this union is disjoint. The set $S^n \setminus h(W)$ is path-connected by lemma 3.6.2. The set $h(W \setminus A)$ is also path connected, since $W \setminus A$ is path-connected (it is homeomorphic to B^n). On the other hand by the Jordan-Brouwer separation theorem the set $S^n \setminus h(A)$ has exactly two path-connected components. It follows that these components are exactly $S^n \setminus h(W)$ and $h(W \setminus A)$. But S^n is locally path-connected, so components of its open subset $S^n \setminus h(A)$ must be open. In particular $h(W \setminus A)$ is open. Hence $Z = W \setminus A$ is an open neighbourhood of x, whose image is also open. □

Inavriance of Domain is also true for subsets of $\mathbb{R}^n$, just like Jordan-Brouwer separation theorem. This time the proof of this fact is elementary - $\mathbb{R}^n$ is (homeomorphic to the) open subset of S^n, so its subsets are open if and only their open in S^n.

Define
$$\mathbb{H}_n = \{x \in \mathbb{R}^n | x_n \geq 0\}.$$
Recall that a non-empty topological space M is called a **topological n-manifold (with boundary)** if
1) M is Hausdorff and
2) M is locally homeomorphic to $\mathbb{R}^n$ or $\mathbb{H}_n$. Precisely these means that every point x of M has a neighbourhood U which is homeomorphic to an open subset of $\mathbb{R}^n$ or open subset of $\mathbb{H}_n$.
This definition is a bit redundant since every open subset of $\mathbb{R}^n$ is clearly homeomorphic to the open subset of $\mathbb{H}_n$. Also one often includes the requirement that M is second countable or paracompact in the definition of a manifold, but we won't need such technical requirements, so we omit them for the sake of simplicity.

Suppose M is a manifold. Any homeomorphism $f\colon U \to f(U) \subset M$, where U is an open subset of $\mathbb{R}^n$ or $\mathbb{H}_n$ and $f(U)$ is open in M is called **a chart** in M.
A point $x \in M$ is called **a boundary point** if there is a chart $f\colon U \to f(U)$ such that U is an open subset of $\mathbb{H}_n$ and $x = f(y)$ for some $y \in \{z \in \mathbb{H}^n \mid z_n = 0\}$. The set of all boundary points of M is denoted by ∂M.
The point $x \in M$ is called **an interior point** if there is a chart $f\colon U \to f(U)$ such that U is an open subset of $\mathbb{R}^n$. The set of all interior points is denoted by $\operatorname{int} M$.
If $\partial M = \emptyset$ (which means that all possible charts of M are defined on the open subsets of $\mathbb{R}^n$), we say that M is a manifold **without boundary**.

Using invariance of domain and other information available to us it is easy to prove the following results concerning manifolds. The proofs are left to the reader (Exercises 3.35 and 3.36).

Lemma 3.6.6. *Suppose M is an n-manifold. Then sets ∂M and* $\operatorname{int} M$ *are disjoint.* $\operatorname{int} M$ *is open in M and itself is an n-manifold without boundary. ∂M is closed in M and is an $(n-1)$-manifold without boundary. No n-manifold is homeomorphic to m-manifold for $m \neq n$.*

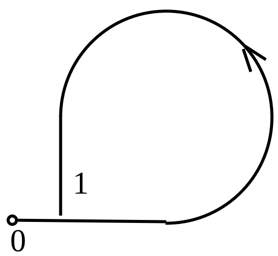
Lemma 3.6.7. *Suppose M is an m-manifold, N is an n-manifold.*
1) If $m > n$ there are no continuous injections $M \to N$.
2) If $m = n$ and M has no boundary, then any continuous injection $f: M \to N$ is an open embedding, i.e. a homeomorphism to the image $f(M)$, which is open in N (and is a subset of $\operatorname{int} M$*).*

Corollary 3.6.8. *Suppose M is a compact n-manifold without boundary and N is a connected n-manifold. If $f: M \to N$ is a continuous injection, then it it a surjective homeomorphism.*

Proof. By the previous lemma $f(M)$ is open in N. On the other hand $f(M)$ is compact, since M is compact, so $f(M)$ is also closed in N. Since N is connected $f(M) = N$. □

Example 3.6.9. *Examples of n-manifolds without boundary include $\mathbb{R}^n$ (and all open subsets of $\mathbb{R}^n$), sphere S^n, projective plane $\mathbb{R}P^n$, torus and Klein bottle, which are 2-manifolds. Mobius band is a 2-manifold with boundary. Closed disk $\overline{B}^n$ is an n-manifold with boundary.*

Example 3.6.10. *The claim 2) of Lemma 3.6.7 is not true if M has boundary. For example consider the mapping $f: [0,1[\to S^1$, $f(t) = e^{2\pi t i}$. Then f is a continuous bijection between two 1-manifolds but it is not homeomorphism. Also if $m < n$ there might be continuous injection from m-manifold M to n-manifold N, which is not embedding, even if M has no boundary. For example let $f:]0,1[\to \mathbb{R}^2$ be a mapping defined as in the picture below.*



Then f is not embedding and the image $f]0,1[$ is not even a manifold

Corollary 3.6.8 shows that if M is compact n-manifold without boundary and N is a connected n-manifold, which is not compact, then M cannot be embedded in N. For example it follows that S^n cannot be embedded in $\mathbb{R}^n$. There is also more precise result known as **the Borsuk-Ulam theorem**, which says that for any mapping $f\colon S^n \to \mathbb{R}^n$ there is $x \in S^n$ such that $f(x) = f(-x)$, but the proof is too difficult for us at this point.

3.7 The degree of a mapping

Suppose $f\colon S^n \to S^n$ is a continuous mapping ($n \geq 1$). Then the induced mapping $f_*\colon H_n(S^n) \to H_n(S^n)$ "looks like" the homomorphism $\mathbb{Z} \to \mathbb{Z}$.
Precisely put $H_n(S^n)$ is a free group on one generator. There are precisely two choices for this generator - if a is a generator, then $-a$ is the only other possibility.
Now there exists unique $m \in \mathbb{Z}$ such that $f_*(a) = ma$. Moreover n does not depend on the choice of the generator, since then $f_*(-a) = -f_*(a) = -ma = m(-a)$. Also in this case for every $x \in H_n(S^n)$ we have

$$f_*(x) = mx$$

and this property characterizes m uniquely.

Definition 3.7.1. *Suppose $f\colon S^n \to S^n$ is a continuous mapping ($n \geq 1$). The unique $m \in \mathbb{Z}$ for which*

$$f_*(x) = mx, x \in H_n(S^n)$$

is called **the degree** *of the mapping f and denoted* $\deg f$.

Let us start by listing the basic properties of the degree

Proposition 3.7.2. *1)* $\deg \mathrm{id} = 1$.
2) $\deg(g \circ f) = \deg g \cdot \deg f$.
3) If $f \simeq g$ are homotopic, then $\deg f = \deg g$.
4) If f is not surjective, then $\deg f = 0$.
5) If f is a homotopy equivalence, then $\deg f = \pm 1$.
6) Suppose $h\colon S^n \to S^n$ is antipodal mapping $h(x) = -x$. Then $\deg h = (-1)^{n+1}$

Proof. Exercise 3.38 (except for property 6, which was earlier exercise 3.28). □

Lemma 3.7.3. *Suppose $f, g \in S^n \to S^n$ are such that $f(x) \neq -g(x)$ for all $x \in S^n$. Then f and g are homotopic. In particular* $\deg f = \deg g$.

Proof. Assumption implies that the interval

$$\{(1-t)f(x) + tg(x) \mid t \in [0,1]\} \subset \mathbb{R}^n$$

from $f(x)$ to $g(x)$ does not contain 0. Hence the mapping $H\colon S^n \times I \to S^n$ defined by

$$H(x,t) = \frac{(1-t)f(x) + tg(x)}{|(1-t)f(x) + tg(x)|}$$

is well-defined and continuous homotopy from f to g. □

Corollary 3.7.4. *Suppose $f\colon S^n \to S^n$, where n is even. Then there is a point $x \in S^n$ such that $f(x) = x$ or $f(x) = -x$.*

Proof. Let us assume that such a point does not exist. Then for the mappings id and h(antipodal mapping, we have $f(x) \neq (-\operatorname{id}(x))$ and $f(x) \neq -h(x)$. By the previous lemma $\deg f = \deg \operatorname{id} = 1$ and at the same time $\deg f = \deg h = (-1)^{n+1} = -1$, since n is even. This is a contradiction □

A continuous mapping $f\colon S^n \to \mathbb{R}^{n+1}$ is called **a tangent vector field** if $\langle x, f(x)\rangle = 0$ for all $x \in S^n$, where $\langle , \rangle$ is standard inner product on $\mathbb{R}^{n+1}$. Geometrically this could be interpreted as the assignment of a tangent vector at every point of S^n, an "arrow" that is parallel to the surface of S^n at this point, in continuous fashion. This explains the name of the following proposition.

Theorem 3.7.5. Hairy Ball's theorem.
Suppose $f\colon S^n \to \mathbb{R}^{n+1}$ is a tangent vector field, where n is even. Then there is a point $x \in S^n$ such that $f(x) = 0$. In other words there is no non-zero vector fields on S^n. Hence if you think of a vector field as "hair" at each point, then there is at least one " bold " spot.

Proof. Suppose $f\colon S^n \to \mathbb{R}^{n+1}$ is a non-zero vector field. Then f defines a mapping $g\colon S^n \to S^n$ by $g(x) = f(x)/|f(x)|$. By the properties of the inner product

$$\langle x, g(x)\rangle = 0$$

for all $x \in S^n$. By the previous corollary there is a point $x \in S^n$ such that $g(x) = x$ or $g(x) = -x$. In both case we obtain

$$\langle x, x\rangle = \langle x, \pm g(x)\rangle = \pm\langle x, g(x)\rangle = 0.$$

By the properties of inner product this implies that $x = 0$. This is impossible, since $x \in S^n$. □

Both corollary 3.7.4 and the Hairy Ball's theorem are not true for odd n (exercise 3.38).

We conclude this section by showing that for every $m, n \in \mathbb{Z}$ there is a mapping $f\colon S^n \to S^n$ with $\deg f = m$.

Proposition 3.7.6. *Let $f_n\colon S^1 \to S^1$ be defined by $p_n(z) = z^n$ (treating $z \in S^1$ as a complex number), $n \in \mathbb{Z}$. Then $\deg p_n = n$.*

Proof. For $n = 0$ the mapping p_0 is a constant mapping, which certainly has degree 0. Also $p_{-1}(x, y) = (x, -y)$ is a reflection along the x-axis, which has degree -1 by exercise 3.27. Since $p_{-n} = p_{-1} \circ p_n$, it is enough to consider the case $n > 0$.
For every $k = 0, \dots, n-1$ let $x_i = e^{2\pi ki/n}$ and define the path $\alpha_k\colon I \to S^1$ by

$$\alpha_k(t) = e^{(1-t)2\pi ki/n + t2\pi(k+1)i/n}.$$

By the exercise 3.29 $x = [\sum_{k=0}^{n-1} \alpha_k]$ is a generator of $H_1(S^1)$ and $x = [\gamma]$, where $\gamma(t) = e^{2\pi it}$. Now $(p_n)_\sharp(\alpha_k) = \gamma$, hence

$$(p_n)_*(x) = \sum_{k=0}^{n-1} [\gamma] = n[\gamma] = nx.$$

□

Let $n > 1$ and suppose $f\colon S^{n-1} \to S^{n-1}$ be a continuous mapping. We define **the suspension** $\Sigma f\colon S^n \to S^n$ of f as follows. Write

$$S^n = \{(x, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid |x|^2 + |t|^2 = 1\}.$$

Assert

$$\Sigma f(x, t) = \begin{cases} (|x| \cdot f(x/|x|), t), & \text{if } x \neq 0, \\ (x, t), & \text{if } x = 0. \end{cases}$$

The geometric idea behind this formula is that for every $c \in [-1, 1]$ the " slice " $\{x \in S^n | x = c\}$ is homeomorphic to S^{n-1} in a natural way, except for extreme cases $c = \pm 1$, where this set reduces to a point (north and south poles of S^n). Using this homeomorphism we define Σf to "look like " f on every slice. North and south poles are fixed points. The verification of continuity of Σf is left as en exercise (3. 39) to the reader.

Proposition 3.7.7. $\deg \Sigma f = \deg f$.

Proof. Recall that $(S^n; B_+, B_-)$ is a proper triad (Exercise 3.33). Notice that $\Sigma f(B_+) \subset B_+$ and $\Sigma f(B_-) \subset B_-$. Also $\Sigma f|S^{n-1} = f$. By the naturality of the reduced Mayer-Vietoris sequence of the proper triad $(S^n; B_+, B_-)$ we obtain a commutative diagram

$$\begin{array}{ccc} H_n(S^n) & \xrightarrow{\partial \cong} & H_{n-1}(S^{n-1}) \\ \downarrow{\scriptstyle (\Sigma f)_*} & & \downarrow{\scriptstyle f_*} \\ H_n(S^n) & \xrightarrow{\partial \cong} & H_{n-1}(S^{n-1}), \end{array}$$

where vertical mappings are isomorphisms. The claim follows. □

As the last application we will prove the fundamental theorem of algebra.

Theorem 3.7.8. *Every non-constant polynomial* $p\colon \mathbb{C} \to \mathbb{C}$ *has at least one root.*

Proof. Suppose p is a non-constant polynomial that does not have roots. We may assume that p is of the form

$$p(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$$

for some $a_0, \ldots, a_{n-1} \in \mathbb{C}$. Since p does not have roots, the mapping $f_r\colon S^1 \to S^1$ defined by

$$f_r(z) = p(rz)/|p(rz)|$$

is well defined and continuous for all $r \geq 0$. Also the mapping $f\colon S^1 \times [0, \infty[\to S^1$,

$$f(z, r) = f_r(z) = p(rz)/|p(rz)|$$

is well-defined and continuous. For every $r > 0$ the restriction $f|S^1 \times [0, r]$ is a homotopy between constant mapping f_0 and f_r. Hence $\deg f_r = \deg f_0 = 0$ for all $r > 0$.

On the other hand let r be any real number such that $r > 1 + |a_0| + |a_a| + \ldots + |a_{n-1}|$. Then for z with $|z| = r$ we have

$$|z^n| = r^n = r \times r^{n-1} > (|a_0| + |a_a| + \ldots + |a_{n-1}|)|z^{n-1}|$$
$$\geq |a_{n-1}z^{n-1}| + \ldots + |a_1 z| + |a_0| \geq |a_{n-1}z^{n-1} + \ldots + a_1 z + a_0|.$$

It follows that for $0 \leq t \leq 1$ the polynomial $p_t(z) = z^n + t(a_{n-1}z^{n-1} + \ldots + a_1 z + a_0)$ has no roots in the set $\{z \mid |z| = r\}$. In particular the homotopy $H\colon S^1 \times [0, 1] \to S^1$ defined by

$$H(z, t) = p_t(rz)/|p_t(rz)|$$

is well-defined. Hence $f_r = H(\cdot, 1)$ is homotopic to $p_n = H(\cdot, 0)$, $p_n(z) = z^n$. By the proposition 3.7.6 it follows that $\deg f_r = \deg p_n = n$.
Hence $n = 0$, so p must be constant polynomial. □

3.8 Exercises

3.8.1 Zeroth homology group, path components and reduced homology

1. Suppose X is a topological space and let $(X_a)_{a\in\mathcal{A}}$ be the set of all path-components of X. Prove that the chain inclusions $(i_\alpha)_\sharp\colon C(X_\alpha) \to C(X)$ induce an isomorphism

$$(i_a)_{a\in\mathcal{A}}\colon \oplus_{a\in\mathcal{A}} C(X_a) \to C(X)$$

of chain complexes.

2. Suppose the chain complex C is a direct sum of complexes $(C_a)_{a\in\mathcal{A}}$. Prove that the inclusion mappings $i_a\colon C_a \to C$ induce a chain isomorphism

$$((_a)_*)_{a\in\mathcal{A}}\colon \oplus_{a\in\mathcal{A}} H_n(C_a) \to H_n(C)$$

for every $n \in \mathbb{N}$.
Deduce the following: Suppose X is a topological space and let $(X_a)_{a\in\mathcal{A}}$ be the set of all path-components of X. Then the inclusions

$$i_a\colon X_a \to X$$

induce an isomorphism

$$((i_a)_*)_{a\in\mathcal{A}}\colon \oplus_{a\in\mathcal{A}} H_n(X_a) \to H_n(X)$$

for every $n \in \mathbb{N}$.

3. Suppose $f\colon X \to Y$ is continuous and X, Y are both path-connected and non-empty. Show that $f_*\colon H_0(X) \to H_0(Y)$ is an isomorphism.

4. Suppose

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & A_{n-1} & \longrightarrow & A_n & \longrightarrow & A_{n+1} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & B_{n-1} & \longrightarrow & B_n & \longrightarrow & B_{n+1} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & C_{n-1} & \longrightarrow & C_n & \longrightarrow & C_{n+1} & \longrightarrow & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\dots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

is a commutative diagram of abelian groups and homomorphisms. Assume that all columns are exact and the middle row is exact. Prove thatthe upper row is exact if and only if lower row is exact. (Hint: all horizontal sequences are chain complexes. Apply the long exact homology sequence).

5. Suppose $x \in X$, where X is a topological space. Prove that

$$H_n(X, x) \cong \tilde{H}_n(X)$$

for all $n \in \mathbb{N}$. (Hint: reduced homology sequence of a pair.)

6. Suppose C is a chain complex with an augmentation ε. Prove that the sequence

$$\dots \longrightarrow C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \dots \longrightarrow C_0 \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0 \longrightarrow \dots$$

is a chain complex C', and $H_n(\widetilde{C}) = H_n(C')$ for all $n \in \mathbb{Z}$. This give another interpretation of reduced homology groups.

7. Calculate the reduced homology groups of two-points discrete space $S^0 = \{a, b\}$ (only the 0th reduced homology is interesting) straight from the definition, i.e. using the reduced singular chain complex. In particular show that $\widetilde{H}_0(S^0) \cong \mathbb{Z}$ and $[a - b]$ is a generator of $\widetilde{H}_0(S^0)$.

3.8.2 Homotopy axiom

8. Consider a prism $\Delta_n \times I$. Denote $v_i = (e_i, 0)$ and v_i', $i \in \{0, \dots, n\}$. Prove that the set $\{v_0, \dots, v_i, v_i', \dots, v_n'\}$, $i = 0, \dots, n$ is independent for every $i = 0, \dots, n$. Show that the simplices obtained so form a simplicial complex, which is a triangulation of the prism $\Delta_n \times I$.

9. Suppose C is a convex subset of a finite-dimensional vector space and $x \in X$. Construct an explicit homotopy equivalence $f\colon (C, \{x\}) \to (\{x\}, \{x\})$.

10. Prove that $\mathbb{R}^n \setminus \{0\}$ has the same homotopy type as S^{n-1} or a punctured ball $\overline{B}^n \setminus \{0\}$.

11. Suppose $f\colon (X, A) \to (Y, B)$ is mapping of pairs. Suppose that $f\colon X \to Y$ as well as $f|A\colon A \to B$ are homotopy equivalences. Prove that

$$f_*\colon H_n(X, A) \to H_n(Y, B)$$

is an isomorphism (Hint: Five Lemma).

12. Prove that the space X is contractible if and only if it has the same homotopy type as a singleton space $\{x\}$ and the pair (X, x) is contractible if and only if it has the same homotopy type as the pair $(\{x\}, \{x\})$. Show that every contractible space is path-connected.

13. Suppose K is a finite Δ-complex. For every geometric n-simplex σ of K choose a point $x_\sigma \in \operatorname{int}\sigma$ and let $U = |K^n| \setminus \{x_\sigma | \sigma \in K_n/\sim\}$. Prove that U is open in K^n and the inclusion $|K^{n-1}| \hookrightarrow U$ is a homotopy equivalence.
Deduce that the inclusion $i\colon (|K^n|, |K^{n-1}|) \to (|K^n|, U)$ induces isomoprhisms in relative homology in all dimensions.

14. Let

$$X = \bigcup_{n \in \mathbb{N}_+} \{1/n\} \times I \cup \{0\} \times I \cup I \times \{0\}$$

and $x_0 = (0, 1)$. Prove that X is contractible, but the pair (X, x_0) is **not** contractible.

15. Suppose C', C, D, D' are chain complexes, $f, g, h\colon C \to D$, $k, m\colon D \to D'$, $l\colon C' \to C$ are chain mappings.
a) Suppose H is chain homotopy from f to g, H' chain homotopy from g to h. Prove that $H+H'$ is a chain homotopy from f to h. Deduce that the relation "f and g are chain homotopic" is an equivalence relation in the set of all chain mappings $C \to D$.
b) Prove that $k \circ H$ is a chain homotopy from $k \circ f$ to $k \circ h$ and $H : l$ is a chain homotopy from $f \circ l$ to $g \circ l$.
c) Suppose H'' is a chain homotopy from k to m. Then $H'' \circ f + m \circ H$ and $k \circ H + H'' \circ g$ are chain homotopies from $k \circ f$ to $m \circ g$.

3.8.3 Excision

16. a) Let $B_+ = \{x \in S^n \mid x_{n+1} \geq 0\}$. Prove that the mapping $f\colon B_+ \to \overline{B}^n$,
$$f(x) = (x_1, \ldots, x_n)$$
is a homeomorphism, which takes e_{n+1} to 0.
b) Show that $U = S \setminus \{e_{n+1}\}$ is homeomorphic to $\mathbb{R}^n$, via stereographic projection through the north pole e_{n+1}. Stereographic projection of the point $y \in U$ is defined to be the unique point in $\mathbb{R}^n \subset \mathbb{R}^{n+1}$ which lies on the line spanned by y and e_{n+1}. Construct the explicit formula for this projection and its inverse.

17. a) Suppose $U \subset \mathbb{R}^n$ is open and $x \in U$. Prove that
$$j_*\colon H_m(U, U \setminus \{x\}) \cong H_m(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$$
for all $m \in \mathbb{N}$. Here j is an obvious inclusion of pairs.
b) Suppose $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are both open and there is a homeomorphism $f\colon U \to V$. Prove that $n = m$.

18. Suppose $f\colon \overline{B}^n \to \overline{B}^n$ is a homeomorphism. Show that f maps interior B^n onto itself and the boundary S^{n-1} also onto itself. (Hint: remove a point).

19. Let X be a non-empty set. Define $C_n(X)$ to be the free abelian group generated on the set X^{n+1} for $n \geq 0$ and $C_n(X) = 0$ for $n < 0$. Prove that the definition
$$\partial(x_0, \ldots, x_n) = \sum_{i=0}^{n} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_n)$$

defines a boundary operator that makes the collection $C(X) = \{C_n(X), \partial\}$ a chain complex. Prove that $C(X)$ has an augmentation $\varepsilon\colon C_0(X) \to \mathbb{Z}$ defined by $\varepsilon(x) = 1$ on generators.

For a fixed $x \in X$ and every $n \geq 0$ define homomorphism $x\colon C_n(X) \to C_{n+1}(X)$ by

$$x(x_0, \ldots, x_n) = (x, x_0, \ldots, x_n).$$

Prove that

$$(\partial_{n+1} x + x\partial_n)(y) = \begin{cases} y, & \text{if } n \neq 0, \\ y - \varepsilon(y)b, & \text{if } n = 0. \end{cases}$$

for all $y \in C(X)$. Deduce that the complex $\widetilde{C}(X)$ is acyclic.

20. Suppose C, D are chain complexes and $f_n, g_n\colon C_n \to D_n$ homomorphisms defined for every $n \in \mathbb{Z}$. Suppose for every $n \in \mathbb{N}$ there exists a homomorphism $H_n\colon C_n \to D_{n+1}$ with the property

$$\partial_{n+1} H_n + H_{n-1}\partial_n = f_n - g_n \text{ for all } n \in \mathbb{Z}.$$

Prove that $f - g = \{f_n - g_n \mid n \in \mathbb{Z}\}$ is a chain mapping.
Deduce that if g is a chain mapping, also f is. In other words **mapping that is homotopic to a chain mapping is a chain mapping itself**.

21. Define a homotopy $H_n\colon C_n(X) \to C_{n+1}X$ by

$$H_n(\sigma) = \sigma_\sharp(H_n(\Delta_n)),$$

where $H_n(\Delta_n)$ is the image of $\mathrm{id}\colon \Delta_n \to \Delta_n$ under $H_n\colon LC_n(\Delta_n) \to LC_{n+1}(\Delta_n) \subset C_n(\Delta_n)$. Prove (using the corresponding property of $H_n\colon LC_n(\Delta_n) \to LC_{n+1}(\Delta_n)$) that H is a chain homotopy between id and barycentric subdivision operator $S\colon C(X) \to C(X)$.

22. a) Suppose D is a convex subset in a finite-dimensional vector space V. Prove that the barycenteric subdivision operator $S\colon LC(D) \to LC(D)$ is a chain mapping straight from the definition, i.e. show that $\partial S = S\partial$, without the use of any homotopy.
b) Do the same for $S\colon C(X) \to C(X)$ for any topological space X.

23. Let

$$B_+ = \{x \in S^n \mid x_{n+1} \geq 0\} \text{ and}$$
$$B_- = \{x \in S^n \mid x_{n+1} \leq 0\} \text{ and .}$$

Use homology and excision axioms to show that the inclusions $i\colon (B_+, S^{n-1}) \to (S^n, B_-)$ and $j\colon (B_-, S^{n-1}) \to (S^n, B_+)$ induce isomorphism in relative homology (for all dimensions).

3.8.4 The equivalence of the simplicial and singular homologies

24. Consider the contractible homotopy $\alpha\colon \Delta_n \times I \to \Delta_n$ defined by

$$\alpha(x,t) = (1-t)x + te_0.$$

Let $\Lambda_n^0 = \bigcup_{i>0} \partial_n^i \Delta_n$. Prove that $\alpha(\Lambda_n^0 \times I) \subset \Lambda_n^0$.

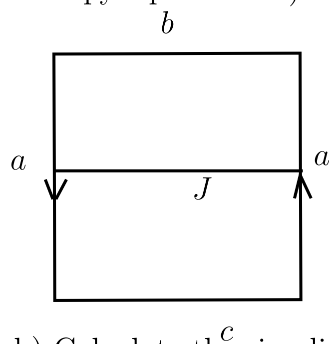
25. Prove that the singular homology **has compact carriers** in the following precise sense.
a) Suppose $x \in H_n(X)$ (X a top. space). Prove that there exists compact $C \subset X$ such that x belongs to the image of

$$i_*\colon H_n(C) \to H_n(X)$$

(where $i\colon C \to X$ inclusion).
b) Suppose $C \subset X$ is compact, $i\colon C \to X$ an inclusion and $x \in H_n(C)$ is such that $i_*(x) = 0 \in H_n(X)$. Prove that there exists a compact $D \subset X$ such that $C \subset D$ and $j_*(x) = 0 \in H_n(D)$, where $j\colon C \to D$ is inclusion.
Also prove a) and b) for reduced homology groups $\tilde{H}_n$.

26. Suppose K is a Δ-complex.
a) Let C be a compact subset of $|K|$. Show that there is a finite subcomplex L of K such that $C \subset L$.
b) Assume the theorem 3.4.3 is true for all finite subcomplexes of K. Prove that $i_*\colon H_n(K) \to H_n(|K|)$ is an isomorphism for all $n \in \mathbb{N}$. (Hint: a) and a previous exercise).

27. a) Prove that the Mobius band X has the same homotopy type as S^1. (Hint: show that inclusion $J \to X$, where J as in the picture is a homotopy equivalence.)



b) Calculate the simplicial homology of the "boundary" in the picture above generated by the 1-simplices a, b, c.

c) Deduce that Mobius band and S^1 are not homeomorphic (remove a point and use b)).

28. a) Let $n > 0, i \in \{1, \ldots, n+1\}$ and let $\iota_i \colon S^n \to S^n$ be defined by $\iota_i(x) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_n + 1)$. Prove that

$$(\iota_i)_*(x) = -x$$

for all $x \in H_n(S^n)$, $i = 1, \ldots, n$, assuming we already know it is true for $i = n+1$. (Hint: use the fact that $\iota_i = f \circ \iota_{n+1} \circ f$ for some homeomorphism f.)
b) Let $h \colon S^n \to S^n$, $h(x) = -x$. Prove that

$$h_*(x) = (-1)^{n+1} x.$$

for all $x \in H_n(S^n)$.

3.8.5 Mayer-Vietoris sequence

29. Suppose $(X; U, V)$ is a proper triad such that $U \cap V \neq \emptyset$. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C(U \cap V) & \xrightarrow{i} & C(U) \oplus C(V) & \xrightarrow{j} & C(U) + C(V) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon \oplus \varepsilon} & & \downarrow{\scriptstyle \varepsilon} & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{i} & \mathbb{Z} \oplus \mathbb{Z} & \xrightarrow{j} & \mathbb{Z} & \longrightarrow & 0,
\end{array}$$

where $\mathbb{Z}$ is considered as a chain complex (with only non-trivial group in dimension 0) and i and j are defined as usual. Show that the diagram is commutative and rows are exact. Conclude that the sequence

$$0 \longrightarrow \widetilde{C}(U \cap V) \xrightarrow{i} \widetilde{C}(U) \oplus \widetilde{C}(V) \xrightarrow{j} \widetilde{C(U) + C(V)} \longrightarrow 0$$

is short exact.

Prove that the inclusion $i\colon \widetilde{C(U) + C(V)} \to \widetilde{C}(X)$ induces isomorphisms in reduced homology (Hint: suitable diagram and the Five Lemma).

Conclude the existence of the reduced Mayer-Vietoris sequence

$$\ldots \longrightarrow \widetilde{H}_{n+1}(X) \xrightarrow{\partial} \widetilde{H}_n(U \cap V) \xrightarrow{i_*} \widetilde{H}_n(U) \oplus \widetilde{H}_n(V) \xrightarrow{j_*} \widetilde{H}_n(X) \xrightarrow{\partial} \widetilde{H}_{n-1}(U \cap V) \longrightarrow \ldots$$

30. Suppose $D = \{0 = t_0 < t_1 < \ldots < t_n = 1\}$ be a finite subdivision of $I = [0, 1]$. Define for every $i = 0, \ldots, n-1$ a path $\alpha_i\colon I \to S^1$ by

$$\alpha_i(t) = \cos(2\pi t_i(1-t) + t2\pi t_{i+1}) + i\sin(2\pi t_i(1-t) + t2\pi t_{i+1}).$$

In other words α_i is an arc that connects $x_i = e^{2\pi t_i}$ and $x_{i+1} = e^{2\pi t_{i+1}}$. Define $\gamma_D \in C_1(S^1)$ as

$$\gamma_D = \sum_{i=0}^{n-1} \alpha_i.$$

Show that γ_D is a cycle. By induction on n prove that $\gamma_D = \gamma$, where $\gamma = \gamma_{D_0}$, $D_=\{0, 1\}$. (Hint: exercise 2.11).

31. Define the equivalence relation $\sim$ on S^n to be the smallest equivalence relation with $x \sim -x$ for all $x \in S^n$ and the equivalence relation $\sim'$ on $\overline{B}^n$ with $x \sim' -x$ for all $x \in S^{n-1}$ Define the mapping $p\colon S^n \to \overline{B}^n/\sim'$ by

$$p(x_1, \ldots, x_{n+1}) = [(x_1, \ldots, x_n)] \text{ if } x_{n+1} \geq 0,$$

$$p(x_1, \ldots, x_{n+1}) = [(-x_1, \ldots, -x_n)] \text{ if } x_{n+1} \leq 0.$$

Prove that p is well-defined, continuous and induces homeomorphism $S^n/\sim \to \overline{B}^n/\sim'$.

32. Suppose K is a simplicial complex and L_1 and L_2 are subcomplexes of K such that $K = L_1 \cup L_2$. Show that $(|K|; |L_1|, |L_2|)$ is a proper triad. (Hint: use the equivalence of simplicial and singular homologies).

33. Use the previous exercise to show that $(S^n; B_+, B_-)$ is a proper triad. Write down the reduced Mayer-Vietoris sequence of this triad and use it to prove that $\widetilde{H}_n(S^n) \cong \widetilde{H}_{n-1}(S^{n-1})$.

3.8.6 Some classical applications

34. Construct the explicit formula for the mapping g defined in the proof of the Brouwer's fixed point theorem (theorem 3.6.1) and show that g is continuous retract $\overline{B}^n \to S^{n-1}$.

35. a) Suppose V is an open subset of $\mathbb{R}^n, n \geq 2$ and $x \in V$. Using excision property show that $H_1(V, V \setminus \{x\}) \cong H_1(\mathbb{R}^n, \mathbb{R}^n \setminus \{x\})$ and deduce that $H_1(V, V \setminus \{x\}) = 0$.
Using this, prove that $V \setminus \{x\}$ is path-connected, if V is path-connected.
b) Suppose $n \geq 2$ and $S \subset \mathbb{R}^n$ is homeomorphic to S^{n-1}.
Prove that $\mathbb{R}^n \setminus S$ has exactly two path components and S is a boundary of both of them, using the corresponging statement for the subsets of S^n.
What happens if $n = 1$?

36. Suppose M is an n-manifold. Prove that
1) The sets ∂M and $\operatorname{int} M$ are disjoint.
2)$\operatorname{int} M$ is open in M and itself is an n-manifold without boundary.
3)∂M is closed in M and is an $n-1$-manifold without boundary.
4) No n-manifold is homeomorphic to m-manifold for $m \neq n$.

37. Suppose M is an m-manifold, N is an n-manifold. Prove that
1) If $m > n$ there are no continuous injections $M \to N$.
2) If $m = n$ and M has no boundary, then any continuous injection $f: M \to N$ is an open embedding, i.e. a homeomorphism to the image $f(M)$, which is open in N (and is a subset of $\operatorname{int} M$).

3.8.7 The degree of a mapping

38. Prove the following claims for the mappings $S^n \to S^n$.
1) $\deg \mathrm{id} = 1$.
2) $\deg(g \circ f) = \deg g \cdot \deg f$.
3) If $f \simeq g$ are homotopic, then $\deg f = \deg g$.
4) If f is not surjective, then $\deg f = 0$.
5) If f is a homotopy equivalence, then $\deg f = \pm 1$.

39. Define $f \colon S^{2n-1} \to S^{2n-1}n$ by

$$f(x_1, x_2, \ldots, x_{2n-1}, x_{2n}) = (x_2, -x_1, \ldots, x_{2n}, -x_{2n-1}).$$

Show that f defines a non-zero vector field.

40. Let $n > 1$ and suppose $f \colon S^{n-1} \to S^{n-1}$ is a continuous mapping. Write $S^n = \{(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R} \mid |x|^2 + |t|^2 = 1\}$ and define $\Sigma f \colon S^n \to S^n$ by the formula

$$\Sigma f(x,t) = \begin{cases} (|x| \cdot f(x/|x|), t), & \text{if } x \neq 0, \\ (x,t), & \text{if } x = 0. \end{cases}$$

Prove that Σf is continuous.

41. Suppose $f \colon S^n \to S^n$ is **even**, i.e. $f(x) = f(-x)$. Prove that $\deg f$ is even integer and if n ie even then $\deg f = 0$.
Give for every $m \in \mathbb{Z}$ an example of an even mapping $f \colon S^1 \to S^1$ with $\deg f = 2m$.

42. a) For every $x \in \overline{B}^n, x \neq 0$ let

$$\alpha(x) = 2\sqrt{\frac{1-|x|}{|x|}}.$$

Define $h \colon \overline{B}^n \to S^n$ by

$$h(x) = \begin{cases} (\alpha x_1, \alpha x_2, \ldots, \alpha x_n, 1 - 2|x|), & \text{if } x \neq 0 \\ e_{n+1} = (0, \ldots, 1) \text{ if } x = 0. \end{cases}$$

Prove that h is a well-defined continuous surjective mapping which restriction to B^n is a homeomorphism to $S^n \setminus \{-e_{n+1}\}$ and which maps S^{n-1} onto $-e_{n+1}$. Deduce that h induces a homeomorphism $\overline{B}^n/S^{n-1} \cong S^n$.
b) Define $f\colon S^n \to S^n$ so that $f|B_+ = h \circ g$, where g is a standard homeomorphism $B_+ \to \overline{B}^n$, $g(x_1, \ldots, x_n, x_{n+1}) = (x_1, \ldots, x_n)$ and $f|B_-$ is a constant mapping that maps everything to the south pole $-e_{n+1}$.
Prove that f is a well-defined continuous mapping and $f(x) \neq -x$ for all $x \in S^n$. Deduce that $\deg f = 1$.