Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 2
19.09-23.09.2011

1. Suppose V is a vector space. Show that the collection $K = \{\sigma_i\}_{i\in I}$ of simplices in V is a simplicial complex if and only if
1) For every simplex σ in K its every face also belongs to K.
2') For every $x \in \bigcup_{i\in I} \sigma_i$ there is a unique $i \in I$ such that x is an interior point of the simplex σ_i.

Solution: Suppose K is a simplicial complex and $x \in \bigcup_{i\in I} \sigma_i$. Then $x \in \sigma_i$ for some $i \in I$. Suppose $v_0, \ldots, v_n$ are vertices of σ_i. Then

$$x = t_0 v_0 + t_1 v_1 + \ldots + t_n v_n,$$

where $t_i \geq 0$ for all $i = 0, \ldots, n$ and $\sum_{i=0}^n t_i = 1$. Define

$$J = \{i \in \{0, \ldots, n\} \mid t_i > 0\}.$$

Then J is non-empty and the simplex σ spanned by the simplices $\{v_i \| \; i \in J\}$ contains x as an interior point.
Let us prove the uniqueness of σ. Suppose $x \operatorname{int} \sigma \cap \operatorname{int} \sigma'$. Then in particular $x \in \sigma \cap \sigma'$. Hence $\sigma'' = \sigma \cap \sigma'$ is non-empty, thus is a face of σ and σ'. On the other hand σ'' intersects the interior of σ (and σ') - at least in x. The only face of a simplex, which intersects the interior of the simplex is the simplex itself. Hence $\sigma = \sigma'' = \sigma$. This proves the uniqueness.

Suppose K is a collection of simplices, that satisfies conditions 1) and 2'). Suppose $\sigma, \sigma' \in K$. Write vertices of σ as $a_0, \ldots, a_k, b_1, \ldots, b_n$ and the vertices of σ' as $a_0, \ldots, a_k, c_1, \ldots, c_m$, where $b_i \neq c_j$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$. Let σ'' be a face of σ spanned by the vertices $\{a_0, \ldots, a_k\}$. By condition 1) $\sigma'' \in K$ Clearly $\sigma \cap \sigma'$ is convex and contains points $a_0, \ldots, a_k$, so

$$\sigma'' \subset \sigma \cap \sigma'.$$

It remains to show the opposite inclusion. Suppose $x \in \sigma \cap \sigma'$. Then

$$x = t_0 a_0 + \ldots + t_k a_k + r_1 b_1 + \ldots + r_n b_n,$$

$$x = t'_0 a_0 + \ldots + t'_k a_k + r'_1 b_1 + \ldots + r'_n b_n,$$

as convex combinations. If some $r_i \neq 0$ or some $r'_i \neq 0$, it follows that x belongs to the interior of two different simplices, which contradicts condition 2)'. Hence $x \in \sigma''$. Thus we have shown that

$$\sigma \cap \sigma'$$

is either empty or is a common face of σ and σ'.

2. Suppose L is a subcomplex of a simplicial complex K. Show that
a) The weak topology on the simplicial complex $|L|$ is the same as the relative topology on $|L|$ induced by the weak topology of $|K|$.
b) $|L|$ is closed in $|K|$.

Solution: First notice the following. Suppose $\sigma \in K$. Then

$$|L| \cap \sigma = \bigcup_{i \in I} \sigma_i,$$

where I is some subset of the set of all faces of σ and $\sigma_i \in L$ for lal $i \in I$. In particular I is finite.
Suppose $C \subset |L|$ is closed in $|L|$ with respect to the weak topology of $|L|$. Let $\sigma \in K$ be an arbitrary simplex. Then

$$C \cap \sigma = (C \cap |L|) \cap \sigma = C \cap (|L| \cap \sigma) = C \cap (\bigcup_{i \in I} \sigma_i) = \bigcup_{i \in I} (C \cap \sigma_i).$$

Every $C \cap \sigma_i$ is closed in σ_i, since C is closed with respect to the weak topology of $|L|$. Moreover σ_i is closed in σ (being its face). Hence $C \cap \sigma_i$ is closed in σ for all $i \in I$. Since I is finite, $C \cap \sigma$ is closed in σ as a finite union of closed sets. Since this is true for every $\sigma \in K$, by the definition of the weak topology C is closed in $|K|$. In particular
1) C is closed with respect to the relative topology on $|L|$ and
2) $|L|$ is closed in $|K|$.

Let $C \subset |L|$ be closed in $|L|$ with respect to the relative topology of $|L|$ as a subset of $|K|$. Since we already know that $|L|$ is closed in $|K|$ this implies that C is closed in $|K|$. By the definition of the weak topology this means that $C \cap \sigma$ is closed in σ for every $\sigma \in K$. In particular this is true for every $\sigma \in L$. Hence C is closed in the weak topology of $|L|$.

3. a) Suppose σ is a simplex in $\mathbb{R}^m$, with vertices $\{v_0, \ldots, v_n\}$. Prove that

$$\operatorname{diam} \sigma = \max\{|v_i - v_j|\},$$

where $|\cdot|$ is a standard norm on $\mathbb{R}^m$.
b) Suppose K is a finite simplicial complex in $\mathbb{R}^m$. Let σ' be a simplex in a first barycentric division $K^{(1)}$, with vertices $\{b(\sigma_0), b(\sigma_1), \ldots, b(\sigma_n)\}$, where $\sigma_0 < \ldots < \sigma_n = \sigma \in K$. Prove that

$$\operatorname{diam} \sigma' \leq \frac{n}{n+1} \operatorname{diam} \sigma.$$

Solution: a) Let

$$M = \max\{|v_i - v_j|\}.$$

It is enough to prove that for all $x, y \in \sigma$

$$|x - y| \leq M.$$

First that us prove this in special case $y = v_j$, $j = 0, \ldots, n$. Now

$$x = t_0 v_0 + \ldots + t_n v_n,$$

where $t_i \geq 0$ for all i and $\sum t_i = 1$. Then

$$|x - v_j| = |\sum t_i v_i - \sum t_i v_j| \leq \sum t_i |v_i - v_j| \leq (\sum t_i) M = M.$$

Next suppose $y = \sum t'_i v_i$. Then

$$|x - y| = |\sum t'_i x - \sum t'_i v_i| \leq \sum t'_i |x - v_i| \leq (\sum t'_i) M = M.$$

b) By a) it is enough to show that

$$|b(\sigma_i) - b(\sigma_j)| \leq \frac{n}{n+1} \operatorname{diam} \sigma$$

for all i, j. We may assume $i < j$. Since $b(\sigma_i), b(\sigma_j) \in \sigma_j$, by the proof of a) we obtain

$$|b(\sigma_i) - b(\sigma_j)| \leq \max\{|b(\sigma_j) - v_k|\},$$

where v_k goes through all the vertices $v_0, \ldots, v_l$ of σ_j. Now

$$|b(\sigma_j) - v_k| = |\sum_{m=0}^{l} 1/(l+1) v_m - \sum 1/(l+1) v_k| = |\sum_{m \neq k} 1/(l+1)(v_m - v_k)| \leq$$

$$\leq \sum_{m \neq k} 1/(l+1) |v_m - v_k| \leq \sum_{m \neq k} 1/(l+1) \operatorname{diam} \sigma = l/(l+1) \operatorname{diam} \sigma.$$

Also $l \leq n$, so

$$l/(l+1) = 1/(1 + 1/l) \leq 1/(1 + 1/n) = n/(n+1).$$

Hence

$$\operatorname{diam} \sigma' \leq \frac{n}{n+1} \operatorname{diam} \sigma.$$

4. Suppose g is a simplicial approximation of the continuous mapping $f\colon |K| \to |K'|$. Show that

$$f(\operatorname{St}(v)) \subset \operatorname{St}(g(v))$$

for every vertex $v \in K$.

Solution: Suppose $x \in St(v)$. Then there exists $\sigma \in K$ such that $x \in \operatorname{int} \sigma$ and v is a vertex of σ. Suppose vertices of σ are $v_0 = v, v_1, \ldots, v_n$. Then there exist $t_i > 0, i = 0, \ldots, n$ such that $\sum t_i = 1$ and

$$x = t_0 v_0 + \ldots + t_n v_n.$$

Since g is simplicial we have

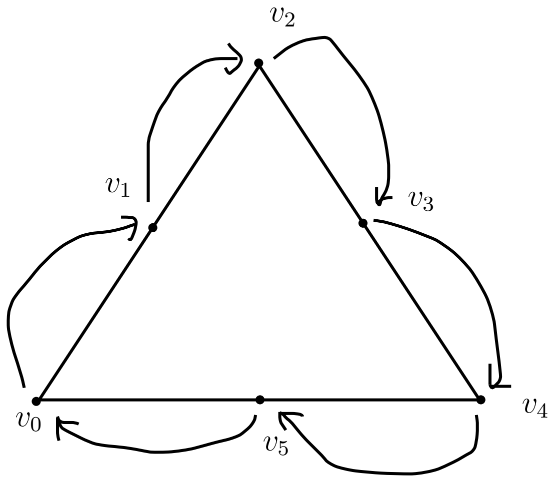
$$g(x) = t_0 g(v_0) + \ldots g(v_n),$$

so $g(x) \in \operatorname{int} \sigma'$, where σ' is a simplex of K' spanned by $g(v_0), \ldots, g(v_n)$. On the other hand suppose $\sigma'' \in K'$ is a unique simplex that contains $f(x)$ in its interior. Then, since g is a simplicial approximation of f, $g(x) \in \sigma''$. Since also $g(x) \in \operatorname{int} \sigma'$, σ' is a face of σ''. In particular $g(v)$ is a vertex of σ''. Hence

$$f(x) \in \operatorname{St}(g(v)).$$

5. Consider the boundary of the equilateral triangle σ as a 2-simplex with vertices v_0, v_2, v_4. For odd $i = 1, \ldots 5$ denote by v_i the barycentre of the 1-simplex $[v_{i-1}, v_{i+1}]$, where we identify $v_6 = v_0$.
Let $K = K(\partial\sigma)$. Let $f\colon |K| \to |K|$ be the unique simplicial mapping $f\colon |K^{(1)}| \to |K^{(1)}|$ defined by $f(v_i) = v_{i+1}$. Prove that as a mapping $f\colon |K| \to |K|$ f does not have a simplicial approximation, but as a mapping $f\colon |K^{(1)}| \to |K|$ f has

exactly 8 simplicial appoximations. List all approximations.

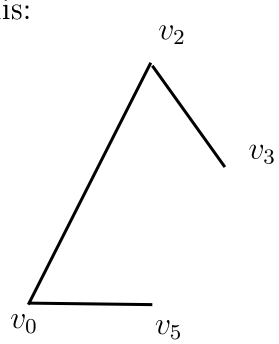


Solution: Suppose K and L are simplicial complexes and $f\colon |K| \to |L|$ is continuous. By the Lemma 1.2.19 f has a simplicial approximation if and only if for every vertex v of K there exists a vertex $v' \in L$ such that

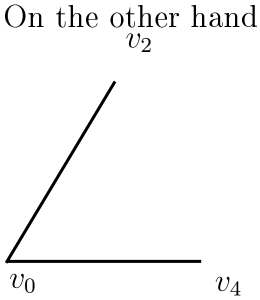
$$f(\mathrm{St}(v)) \subset \mathrm{St}(v').$$

Moreover any choice of such $v' = g(v)$ for every $v \in K$ defines a unique simplicial approximation of f.

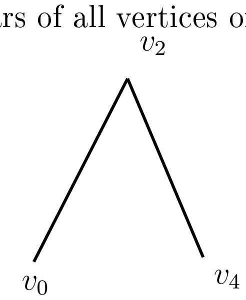
First let us consider f as a mapping $|K| \to |K|$. Now $f(\mathrm{St}(v_0))$ looks like this:



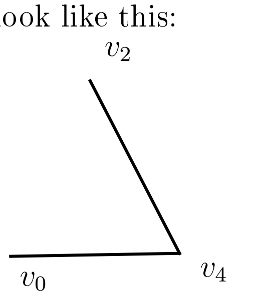
On the other hand stars of all vertices of K look like this:



Star of v_0



Star of v_2



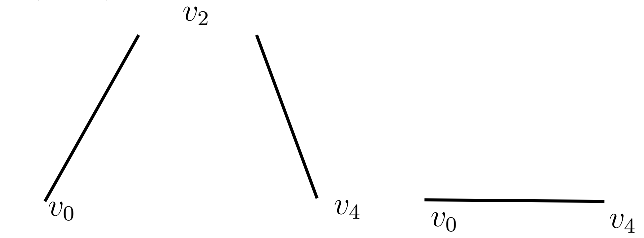
Star of v_4

So one sees immediately, that no vertex $v \in K$ has the property

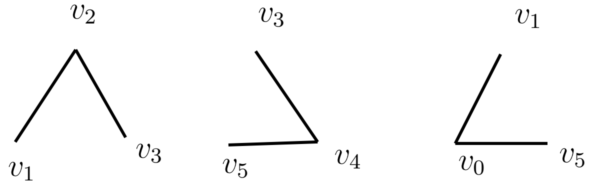
$$f(\mathrm{St}(v_0)) \subset \mathrm{St}(v).$$

In particular f does not have a simplicial approximation.

Now let us consider f as mapping $|K^{(1)}| \to |K|$. The stars of the vertices of $|K|$ are already drawn above. Let us draw the sets $f(\mathrm{St}(v)$ for all vertices v of $|K^{(1)}|$.



$f(\mathrm{St}(v_0))$ $f(\mathrm{St}(v_2))$ $f(\mathrm{St}(v_4))$



$f(\mathrm{St}(v_1))$ $f(\mathrm{St}(v_3))$ $f(\mathrm{St}(v_5))$

We see immediately that for $v = v_0, v_2, v_4$ there are exactly two choices of a vertex $v' \in K$ such that

$$f(\mathrm{St}(v)) \subset \mathrm{St}(v').$$

For instance for v_0 we can choose $v' = v_0$ or $v' = v_2$. On the other hand for $v = v_1, v_3, v_5$ there is only one choice. This implies that there are exactly $2 \cdot 2 \cdot 2 = 8$ simplicial approximations g. We have

$$g(v_i) = v_{i+1}(\text{ mod } 6) \text{ for odd } i,$$

$$g(v_i) \in \{v_i, v_{i+2}\}(\text{ mod } 6 \text{ for even } i.$$

6. a) Suppose $m \in \mathbb{N}$. Let K be a finite m-dimensional simplicial complex and K' be a simplicial complex whose dimension is $> m$. Show that every continuous mapping $f\colon |K| \to |K'|$ is homotopic to a mapping, which is not surjective (Hint: simplicial approximation).
b) Suppose $m < n$. Prove that any continuous mapping $f\colon S^m \to S^n$ is homotopic to a constant mapping.

Solution: a)Suppose $f\colon |K| \to |K'|$ is continuous. By the Simplicial Approximation Theorem f is homotopic to a simplicial mapping $g\colon |K|^{(n)} \to |K|$ for some $n \in \mathbb{N}$. Now $|K|^{(n)}$ is also m-dimensional. Since g is simplicial it maps k-simplex to a simplex, whose dimension is $\leq k$, for all $k \in \mathbb{N}$. In particular, since $|K|^{(n)}$ is m-dimensional it follows that $g(|K|^{(n)}) \subset |K'|^m \neq |K'|$. Hence g is not surjective.
b) S^m is a polyhedron of a finite m-dimensional complex, and S^n is a polyhedron of a complex with dimension $n > m$. Hence by a) a continuous mapping $f\colon S^m \to S^n$ is homotopic to a mapping $g\colon S^m \to S^n$, which is not surjective. Hence there exists $y \in S^n$ such that $g(S^m) \subset S^n \setminus \{y\} = X$. It is

a well-known fact that X is homeomorphic to $\mathbb{R}^n$, in particular contractible to a point. Hence g is homotopic to a constant mapping.

7. Suppose $x \in |K|$.
a)Define $L = \{\sigma \in K | x \notin \sigma\}$. Show that L is a simplicial complex and

$$|K| \setminus |L| = \mathrm{St}(x).$$

Conclude that $\mathrm{St}(x)$ is an open neighbourhood of x in $|K|$.
b)Suppose $x \in |K|$ and all the vertices of $\mathrm{car}(x)$ are $v_0, \ldots, v_n$.
Prove that

$$\mathrm{St}(x) = \bigcup\{\mathrm{int}\,\sigma \mid \mathrm{car}(x) < \sigma\} = \bigcup\{\mathrm{int}\,\sigma \mid v_0, \ldots, v_n \text{ are vertices of } \sigma\}.$$

and

$$\mathrm{St}(x) = \bigcap_{i=0}^{n} \mathrm{St}(v_i).$$

Solution: a) L is clearly closed under faces, so is a simplicial subcomplex of K. Let us prove that

$$|K| \setminus |L| = \mathrm{St}(x).$$

Suppose $y \in |K|$ and let $\sigma \in K$ be the unique simplex such that $y \in \ Int\sigma$. Then $y \in \mathrm{St}(x)$ if and only if $x \in \sigma$, which is true if and only if $\sigma \notin L$. Since σ is a carrier of y and L is a subcomplex the condition $\sigma \notin L$ is equivalent to $y \notin |L|$.
By exercise 2) $|L|$ is closed, hence $|K| \setminus |L|$ is open. Thus $\mathrm{St}(x)$ is an open neighbourhood of x in $|K|$.

b) If $x \in \sigma$, where $\sigma \in K$, then $\mathrm{car}(x)$ must be a face of σ, which proves that

$$\mathrm{St}(x) = \bigcup\{\mathrm{int}\,\sigma \mid \mathrm{car}(x) < \sigma\}.$$

Now it is clear that $\mathrm{car}(x) < \sigma$ if and only if $v_0, \ldots, v_n$ are vertices of σ. By applying this result to every vertex v_i we obtain

$$\mathrm{St}(v_i) = \bigcup\{\mathrm{int}\,\sigma \| \ v_i \in \sigma, \}$$

so

$$\mathrm{St}(x) = \bigcap_{i=0}^{n} \mathrm{St}(v_i).$$