Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 1

1. Consider the pairs $(V, \{v_1, \ldots, v_n\})$, where $V$ is finite-dimensional vector space and $\{v_1, \ldots, v_n\}$ is a fixed basis of $V$. Thus for every $n \in \mathbb{N}$ the pair $(\mathbb{R}^n, \{e_1, \ldots, e_n\})$ is an example of such pair. Moreover for every pair $(V, \{v_1, \ldots, v_n\})$ there is a unique linear bijection $f: V \to \mathbb{R}^n$ such that $f(v_i) = e_i$ for all $i \in \{1, \ldots, n\}$.
   a) Assign to a pair $(V, \{v_1, \ldots, v_n\})$ unique topology such that $f$ as above is a homeomorphism. Prove that $+: V \times V \to V$ and $\cdot: \mathbb{R} \times V \to V$ are continuous with respect to this topology.
   Suppose $(W, \{w_1, \ldots, w_m\})$ is another pair and $l: V \to W$ is linear. Deduce that $l$ is continuous.
   b) Deduce that the topology so assigned to $V$ does not depend on the chosen basis $\{v_1, \ldots, v_n\}$) (apply a) to the identity mapping).

2. Suppose $A \subset V$ is a non-empty subset. Prove that $A$ is affine if and only if there is $v \in V$ and a linear subspace $W \subset V$ such that $A = x + W$. Moreover show that in this case $W$ is unique.

3. a) Show that an affine/convex set $A$ is closed under affine/closed combinations. In other words prove that if $a_1, \ldots, a_n \in A$, $r_1, \ldots, r_n \in \mathbb{R}$, $r_1 + \ldots + r_n = 1$ and in convex case also $r_i \geq 0$ for all $i = 1, \ldots, n$, then
$$r_1 a_1 + \ldots + r_n a_n = x \in A.$$
   b) Suppose $A \subset V$. Prove that
$$\operatorname{aff}(A) = \{r_1 a_1 + \ldots + r_n a_n \mid a_i \in A, r_1 + \ldots + r_n = 1\},$$
$$\operatorname{conv}(A) = \{r_1 a_1 + \ldots + r_n a_n \mid a_i \in A, r_i \geq 0, r_1 + \ldots + r_n = 1\}.$$
   c) Suppose $f: C \to C'$ is a convex mapping between convex sets. Prove that
$$f(r_1 a_1 + \ldots + r_n a_n) = r_1 f(a_1) + \ldots + r_n f(a_n),$$
   if $a_1, \ldots, a_n \in A$, $r_1, \ldots, r_n \in \mathbb{R}$, $r_1 + \ldots + r_n = 1$ and $r_i \geq 0$ for all $i = 1, \ldots, n$.

4. Prove that the set of vertices of a simplex is uniquely determined by the simplex. (Hint: show that a point is not a vertex if and only if it is a midpoint of an interval contained entirely in the simplex).

5. Let $V$ be a finite-dimensional vector space.
   a) Suppose $A \subset V$ and $\{v_0, \ldots, v_n\}$ is a maximal (with respect to inclusion) affinely independent subset of $A$. Prove that $\operatorname{aff}(A) = \operatorname{aff}(\{v_0, \ldots, v_n\})$.
   b) Suppose $C \subset V$ is convex and non-empty. Prove that $C$ has a non-empty interior with respect to $\operatorname{aff}(C)$. (Hint: use a) and notice that simplex spanned by $\{v_0, \ldots, v_n\}$ is a subset of $C$.)

( Exercises continue on the other side!)

6. Show that the standard $n$-simplices defined by

$$\Delta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i, \sum_{i=1}^{n} x_i \leq 1\},$$

$$\Delta'_n = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0 \text{ for all } i, \sum_{i=0}^{n} x_i = 1\}$$

are compact Hausdorff spaces (as subsets of Euclidean spaces).

7. Suppose $C \subset \mathbb{R}^n$ is a closed bounded convex set and 0 is the interior point of $C$. Let $f\colon \partial C \to S^{n-1}$, $f(x) = x/|x|$ and assume known that $f$ is a homeomorphism.
Prove that $G\colon \overline{B}^n \to C$ defined by

$$G(t) = \begin{cases} |t| \cdot \left(f^{-1} \frac{t}{|t|}\right) \text{ if } t \neq 0, \\ 0, \text{ if } t = 0 \end{cases}$$

is a homeomorphism.

Bonus points for the exercises: 25% - 1 point, 40% - 2 points, 50% - 3 points, 60% - 4 points, 75% - 5 points.