Matematiikan ja tilastotieteen laitos
Introduction to Algebraic Topology
Fall 2011
Exercise 1

1. Consider the pairs $(V, \{v_1, \ldots, v_n\})$, where V is finite-dimensional vector space and $\{v_1, \ldots, v_n\}$ is a fixed basis of V. Thus for every $n \in \mathbb{N}$ the pair $(\mathbb{R}^n, \{e_1, \ldots, e_n\})$ is an example of such pair. Moreover for every pair $(V, \{v_1, \ldots, v_n\})$ there is a unique linear bijection $f\colon V \to \mathbb{R}^n$ such that $f(v_i) = e_i$ for all $i \in \{1, \ldots, n\}$.
a) Assign to a pair $(V, \{v_1, \ldots, v_n\})$ unique topology such that f as above is a homeomorphism. Prove that $+\colon V \times V \to V$ and $\cdot\colon \mathbb{R} \times V \to V$ are continuous with respect to this topology.
Suppose $(W, \{w_1, \ldots, w_m\})$ is another pair and $l\colon V \to W$ is linear. Deduce that l is continuous.
b) Deduce that the topology so assigned to V does not depend on the chosen basis $\{v_1, \ldots, v_n\})$ (apply a) to the identity mapping).

Solution: a) Since f is a bijection, there is precisely one way to define a topology in V such that f will become a homeomorphism - define $U \subset V$ to be open if and only if $f(U)$ is open in $\mathbb{R}^n$.
Since f is linear the following diagrams commute

$$\begin{array}{ccc} V \times V & \xrightarrow{+} & V \\ \downarrow{\scriptstyle f\times f} & & \downarrow{\scriptstyle f} \\ \mathbb{R}^n \times \mathbb{R}^n & \xrightarrow{+} & \mathbb{R}^n \end{array}$$

$$\begin{array}{ccc} \mathbb{R} \times V & \xrightarrow{\cdot} & V \\ \downarrow{\scriptstyle \mathrm{id}\times f} & & \downarrow{\scriptstyle f} \\ \mathbb{R} \times \mathbb{R}^n & \xrightarrow{\cdot} & \mathbb{R}^n. \end{array}$$

Since f is a homeomorphism and algebraic operations $+$ and $\cdot$ are continuous in $\mathbb{R}^n$, it follows that they are continuous in V.
Suppose $l\colon V \to W$ is linear. Denote by $f'\colon W \to \mathbb{R}^m$ the corresponding linear bijection that defines topology in W. Then $l' = f' \circ l \circ f^{-1}\colon \mathbb{R}^n \to \mathbb{R}^m$ is linear. It is well-known fact that linear mappings between Euclidean spaces are continuous (see Topology I). Hence l' is continuous, thus also $l = f'^{-1} \circ l' \circ f$ is continuous.

b)Suppose V has two topologies defined as above using different bases. The identity mapping $\mathrm{id}\colon V \to V$ is linear, hence continuous regardless of which topologies we use in the image space and in the domain space. This implies in particular that it is a homemorphism in any possible case, thus the topologies must be the same .

2. Suppose $A \subset V$ is a non-empty subset. Prove that A is affine if and only if there is $v \in V$ and a linear subspace $W \subset V$ such that $A = x + W$. Moreover

show that in this case W is unique.

Solution: Suppose $A = v + W$, where $v \in V$ and W is a linear subspace. Let $x = v + w$, $x' = v + W' \in A$, $t \in \mathbb{R}$. Then

$$(1-t)x + tx' = (1-t)(v+w) + t(v+w') = v + (1-t)w + tw' \in v + W,$$

since W is closed under scalar multiplication and addition. Hence A is affine Incidently this also proves that any translation of an affine set is affine.

Conversely suppose $A \neq \emptyset$ is affine. Fix $v \in A$ and define

$$W = A - v = \{a - v \| \ a \in A\}.$$

Then $A = v + W$. It remains to show that W is linear. Suppose $t \in \mathbb{R}$ and $a \in A$. Then

$$t(a-v) + v = (1-t)v + ta \in A,$$

since A is affine, hence $t(a-v) = (t(a-v)+v) - v \in A - v \in W$. Hence W is closed under scalar multiplication. Suppose $x = a - v, x' = a' - v \in W$. Then

$$(x+y)/2 = (a+a')/2 - v \in A - v = W,$$

since A is convex. Since we already know that W is closed under scalar multiplication, it follows that

$$x + y = 2 \cdot (x+y)/2 \in W.$$

We have shown that W is a linear subspace.

Another way to prove the claim is to show generally that affine sets are invariant under translations (we already sort of it did above) and to prove that a subset of V is a linear subspace if and only if it is affine and contains 0. We leave it to the reader to try this path of solution.

It remains to show the uniqueness. Suppose $A = W + v = W' + v'$, where W, W' are linear subspaces. Then

$$W = W' + (v' - v),$$

so in particular ($0 \in W'$!) it follows that $v' - v \in W$. Moreover thus we obtain

$$W' = W - (v' - v) \subset W,$$

since W is closed under substraction. By the symmetry also $W \subset W'$.

3. a) Show that an affine/convex set A is closed under affine/closed combinations. In other words prove that if $a_1, \ldots, a_n \in A$, $r_1, \ldots, r_n \in \mathbb{R}$, $r_1 + \ldots + r_n = 1$ and in convex case also $r_i \geq 0$ for all $i = 1, \ldots, n$, then

$$r_1 a_1 + \ldots + r_n a_n = x \in A.$$

b) Suppose $A \subset V$. Prove that

$$\text{aff}(A) = \{r_1 a_1 + \ldots + r_n a_n \mid a_i \in A, r_1 + \ldots + r_n = 1\},$$

$$\text{conv}(A) = \{r_1 a_1 + \ldots + r_n a_n \mid a_i \in A, r_i \geq 0, r_1 + \ldots + r_n = 1\}.$$

c) Suppose $f\colon C \to C'$ is an affine mapping between convex sets. Prove that

$$f(r_1 a_1 + \ldots + r_n a_n) = r_1 f(a_1) + \ldots + r_n f(a_n),$$

if $a_1, \ldots, a_n \in A$, $r_1, \ldots, r_n \in \mathbb{R}$, $r_1 + \ldots + r_n = 1$ and $r_i \geq 0$ for all $i = 1, \ldots, n$.

Solution: a) We prove the claim in affine case, leaving the similar convex case to the reader. Suppose A is affine $a_1, \ldots, a_n \in A$, $r_1, \ldots, r_n \in \mathbb{R}$, $r_1 + \ldots + r_n = 1$. We prove by induction on n that $x = r_1 a_1 + \ldots + r_n a_n \in A$. In case $n = 1$ there is nothing to prove. Suppose the claim is true for $n - 1$, $n \geq 2$. Since $r_1 + \ldots + r_n = 1$ it follows that there is an index $i = 1, \ldots, n$ such that $r_i \neq 1$. We may assume that $i = n$. Let $r = r_1 + \ldots + r_{n-1} = 1 - r_n \neq 0$. Define $r'_i = r_i / r$ for $i = 1, \ldots, n-1$. Then $\sum_{i=1}^n r'_1 + \ldots + r'_{n-1} = 1$, hence by inductive assumption

$$y = r'_1 x_1 + \ldots + r'_{n-1} x_{n-1} \in A.$$

Since $x = (1 - r_n) y + r_n x_n$, it follows that $x \in A$ by the very definition of affine set.

b) Suppose $A \subset B$, where B is affine. Then by a) B contains the set

$$C = \{ r_1 a_1 + \ldots + r_n a_n \mid a_i \in A, r_1 + \ldots + r_n = 1 \}$$

Clearly $A \subset C$. It remains to show that C is affine. This is an easy calculation and is skipped. The convex case is similar.

c) Induction on n. Again in case $n = 1$ there is nothing to prove. Suppose the claim is true for $n - 1$, $n \geq 2$. If $r_n = 0$ or $r_n = 1$ there is nothing to prove. In the opposite case define $r' = 1 - r_n = r_1 + \ldots + r_{n-1}$ and $r'_i = r_i / r', i = 1, \ldots, n-1$ as above. Let $y = r'_1 x_1 + \ldots + r'_{n-1} x_{n-1} \in A$. Then $x = (1 - r_n) y + r_n x_n$. By the definition of affine mapping and inductive assumption we obtain

$$f(x) = (1 - r_n) f(y) + r_n f(x_n) = r'(r'_1 f(x_1) + \ldots + r'_{n-1} f(x_{n-1})) + r_n f(x_n) =$$
$$= r_1 f(x_1) + \ldots + r_n f(x_n).$$

4. Prove that the set of vertices of a simplex is uniquely determined by the simplex. (Hint: show that a point is not a vertex if and only if it is a midpoint of an interval contained entirely in the simplex).

Solution: As the hint suggests we prove that the set of vertices of a simplex σ coincides with the set of all points of σ which are not midpoints of an interval contained entirely in the simplex. Since this condition depends only on the set σ itself, this implies the claim.

Let $\{v_0, \ldots, v_n\}$ be vertices of σ. Suppose first $x \in \sigma$ is not a vertex point. Then $x = t_0 v_0 + \ldots + t_n v_n$, where $t_i > 0$ and $t_j > 0$ for at least two distinct indices i, j, $i < j$. Let $\varepsilon > 0$ be such that $t_i - \varepsilon > 0$, $t_j - \varepsilon > 0$. Define

$$y = t_0 v_0 + t_1 v_1 + \ldots + (t_i + \varepsilon) v_i + \ldots + (t_j - \varepsilon) v_j + \ldots + t_n v_n,$$

$$Z = t_0 v_0 + t_1 v_1 + \ldots + (t_i - \varepsilon) v_i + \ldots + (t_j + \varepsilon) v_j + \ldots + t_n v_n.$$

Then $y, z \in \sigma$, $y \neq z$ and $x = (y + z)/2$.

Suppose conversely $x = (y + z)/2$, where $y = t_0 v_0 + \ldots + t_n v_n, z = t'_0 v_0 + \ldots + t'_n v_n \in \sigma$, $y \neq z$. Then there exists an index $i = 0, \ldots, n$ such that

$t_i \neq t'_i$, so in particular at least one of the numbers t_i, t'_i is not equal 0 and consequently $(t_i + t'_i)/2 > 0$. Since

$$t_i = 1 - \sum_{j \neq i} t_j,$$

$$t'_i = 1 - \sum_{j \neq i} t'_j,$$

it follows that there must also be $j \neq i$ such that $t_j \neq t'_j$ (otherwise $t_i = t'_i$). As above we conclude that $(t_j + t'_j)/2 > 0$. It follows that the midpoint

$$x = (y + z)/2$$

of the interval $[y, z]$ has at least two coefficients which differ from zero, hence cannot be a vertex of a simplex σ.

5. Let V be a finite-dimensional vector space.
a) Suppose $A \subset V$ and $\{v_0, \ldots, v_n\}$ is a maximal (with respect to inclusion) affinely independent subset of A. Prove that $\mathrm{aff}(A) = \mathrm{aff}(\{v_0, \ldots, v_n\})$.
b) Suppose $C \subset V$ is convex and non-empty. Prove that C has a non-empty interior with respect to $\mathrm{aff}(C)$. (Hint: use a) and notice that the simplex spanned by $\{v_0, \ldots, v_n\}$ is a subset of C.)

Solution: a) Since $\mathrm{aff}(A)$ is an affine set that contains points $v_0, \ldots, v_n$, it follows that

$$\mathrm{aff}(\{v_0, \ldots, v_n\}) \subset \mathrm{aff}(A).$$

To prove the converse inclusion it is enough to prove that $A \subset \mathrm{aff}(\{v_0, \ldots, v_n\})$. Let us make counter assumption that there is $x \in A$ such that $x \notin \mathrm{aff}(\{v_0, \ldots, v_n\})$. We will prove that then the set $\{x, v_0, \ldots, v_n\}$ is affinely independent, which contradicts the maximality assumptions.
Suppose

$$r_0 v_0 + \ldots + r_n v_n + rx = 0, \text{ where } r_0 + \ldots + r_n + r = 0.$$

We must show that $r_0 = r_1 = \ldots = r_n = r = 0$. If $r = 0$, we are done, since $\{v_0, \ldots, v_n\}$ is already known to be independent. Suppose $r \neq 0$. Then

$$x = (-r_0/r)v_0 + \ldots + (-r_n/r)v_n,$$

where $(-r_0/r) + \ldots + (-r_n/r) = 1$. Hence the right side of the equation is affine combination, which shows that $x \in \mathrm{aff}(\{v_0, \ldots, v_n\})$. This contradicts the choice of x.

b)Since V is finite-dimensional, C cannot contain arbitrary big affinely independent subsets. Hence there exists a maximal affinely independent subset $\{v_0, \ldots, v_n\}$ of C. By a)

$$W = \mathrm{aff}(C) = \mathrm{aff}(\{v_0, \ldots, v_n\}).$$

There exists unique affine mapping $g\colon W \to \mathbb{R}^n$ such that $f(v_i) = e_i$, $i = 0, \ldots, n$ (make sure that g exists!). Moreover such g is then a homeomorphism. Hence it is enough to show that $g(C)$ has interior points. But $g(C)$ is a convex subset of $\mathbb{R}^n$, that contains points $e_0, \dots, e_n$, hence also contains the standard n-simplex Δ_n that they span. The interior

$$int\Delta_n = \{(x_1, \dots, x_n) \| \; x_i > 0, \sum_{i=0}^{n} x_i < 1, x_i > 0 \text{ for all } i\}$$

of Δ_n is clearly a non-empty open subset of $\mathbb{R}^n$. Since it is a subset of $g(C)$ we are done.

6. Show that the standard n-simplices defined by

$$\Delta_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i, \sum_{i=1}^{n} x_i \leq 1\},$$

$$\Delta_n' = \{(x_0, \dots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geq 0 \text{ for all } i, \sum_{i=0}^{n} x_i = 1\}$$

are compact Hausdorff spaces (as subsets of Euclidean spaces).
Solution: We give the proof for Δ_n, the other one being similar. It is enough to show that Δ_n is closed and bounded in $\mathbb{R}^n$. It is closed as a finite intersection of sets

$$A_i = \{x \in \mathbb{R}^n \| \; x_i \geq 0\},$$
$$B = \{x \in \mathbb{R}^n \| \; \sum_{i=1}^{n} x_i \leq 1\},$$

which are easily seen to be closed as an inverse image of closed sets under some obvious continuous mappings $\mathbb{R}^n \to \mathbb{R}$.
Suppose $x \in \Delta_n$. Then $|x_i| \leq 1$ for all $i = 1, \dots, n$, hence

$$|x|^2 = \sum_{i=1}^{n} x_i^2 \leq \sum_{i=1}^{n} 1 = n.$$

We conlude that Δ_n is also bounded.

7. Suppose $C \subset \mathbb{R}^n$ is a closed bounded convex set and 0 is the interior point of C. Let $f \colon \partial C \to S^{n-1}$, $f(x) = x/|x|$ and assume known that f is a homeomorphism.
Prove that $G \colon \overline{B}^n \to C$ defined by

$$G(t) = \begin{cases} |t| \cdot \left(f^{-1} \frac{t}{|t|}\right) \text{ if } t \neq 0, \\ 0, \text{ if } t = 0 \end{cases}$$

is a homeomorphism.

Solution: Let us first prove that G is a bijection. First notice that $G(t) = 0$ if and only if $t = 0$. If $t, t' \neq 0$ and $G(t) = G(t')$, then in particular

$$f^{-1} \frac{t}{|t|} / |f^{-1} \frac{t}{|t|}| = G(t)/|G(t)| = G(t')/|G(t')| = f^{-1} \frac{t}{|t'|} / |f^{-1} \frac{t}{,} |t'||.$$

Letting $a = f^{-1} \frac{t}{|t|}$, $b = f^{-1} \frac{t}{,} |t'|$ we see that $f(a) = f(b)$. Hence $\frac{t}{|t|} = f(a) = \frac{t}{,} |t'| = g(b)$. Since

$$|t| \frac{t}{|t|} = G(t) = G(t') = |t'| \frac{t}{,} |t'| = |t'| \frac{t}{|t|}$$

and $\frac{t}{|t|} \neq 0$, this implies that $|t| = |t'|$. Hence

$$t = |t| \cdot (t/|t|) = |t'| \cdot (t'/|t'|) = t'$$