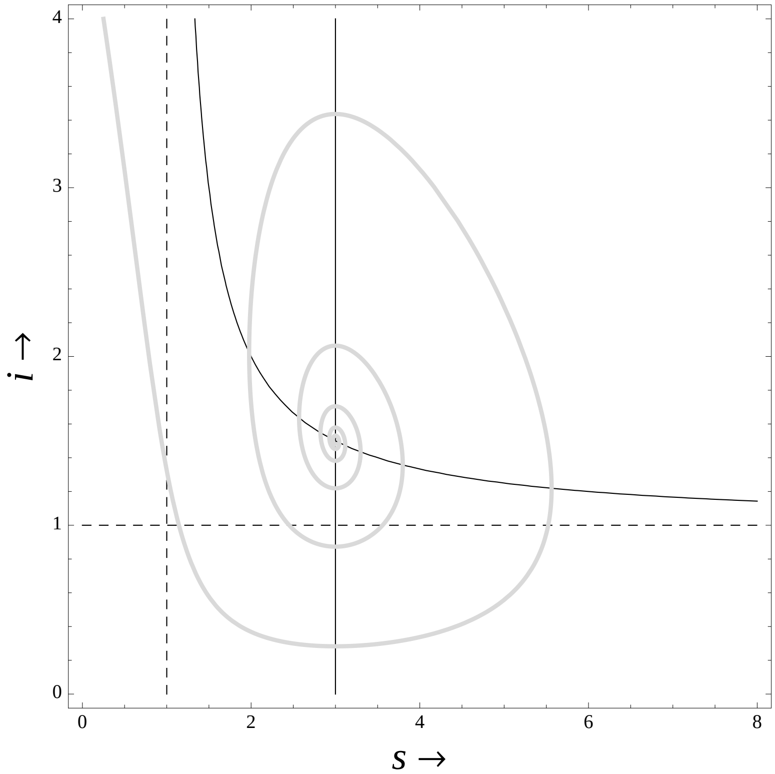
SIS model

Deterministic

```
ds := + (λ - ν) s - β s i + γ i;
di := - (α + ν) i + β s i - γ i;
```

```
λ = 2; ν = 1; α = 1; β = 1; γ = 1;
```

```
sol = NDSolve[{s′[t] == (λ - ν) s[t] - β s[t] i[t] + γ i[t],
    i′[t] == - (α + ν) i[t] + β s[t] i[t] - γ i[t], s[0] == .25, i[0] == 4}, {s, i}, {t, 0, 20}];
obp = ParametricPlot[Evaluate[{s[t], i[t]} /. sol], {t, 0, 20},
   PlotStyle → {LightGray, Thick}];
isp = ContourPlot[{ds, di}, {s, 0, 8}, {i, 0, 4}, ContourStyle → {{Black}}];
ver = Graphics[{Dashed, Line[{{γ / β, 0}, {γ / β, 4}}]}];
hor = Graphics[{Dashed, Line[{{0, (λ - ν) / β}, {8, (λ - ν) / β}}]}];
det = Show[isp, ver, hor, obp, FrameLabel → {"s  →", "i  →"}, ImageSize → Medium]
Clear[λ, ν, α, β, γ];
```



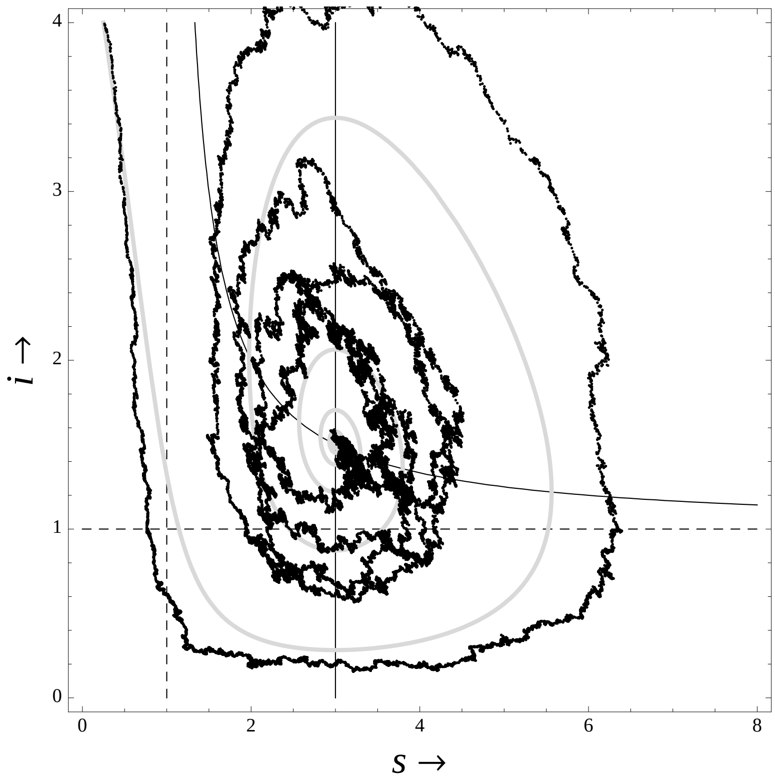
Stochastic

```
μs := + (λ - ν) s - β s i + γ i;
μi := - (α + ν) i + β s i - γ i;

σss := + (λ + ν) s + β s i + γ i;
σsi := - β s i - γ i;
σis := - β s i - γ i;
σii := + (α + ν) i + β s i + γ i;

<< MultivariateStatistics`;
```

```
λ = 2; ν = 1; α = 1; β = 1; γ = 1; ϵ = .01;

dt = 0.0005; tmax = 20;

s = .25; i = 4; t = 0; dat = {};
```

```
While[t ≤ tmax,

   {ds, di} =
    {μs, μi} dt +
     RandomReal[MultinormalDistribution[{0, 0}, ϵ {{σss, σsi}, {σis, σii}} dt]];

   s = s + ds; i = i + di; t = t + dt;
   dat = Join[dat, {{s, i}}];
  ];
```

```
sto = ListPlot[dat, Joined → False, PlotStyle → {Black, PointSize[.001]},
    PlotRange → All, MaxPlotPoints → Infinity, ImageSize → Medium];
Show[det, sto]
Clear[λ, ν, α, β, γ, ϵ, s, i];
```



Cross-covariance OU approximation

```
(* deterministic equilibrium *)
eq = Solve[{μs == 0, μi == 0}, {s, i}] // Last
```

$$\left\{i \to \frac{(\lambda-\nu)\ (\alpha+\gamma+\nu)}{\beta\ (\alpha+\nu)},\ s \to \frac{\alpha+\gamma+\nu}{\beta}\right\}$$

```
A = D[{μs, μi}, {{s, i}}] /. eq // Simplify;
% // MatrixForm
```

$$\begin{pmatrix} \frac{\gamma\ (-\lambda+\nu)}{\alpha+\nu} & -\alpha-\nu \\ \frac{(\lambda-\nu)\ (\alpha+\gamma+\nu)}{\alpha+\nu} & 0 \end{pmatrix}$$

```
B2 = ϵ {{σss, σsi}, {σis, σii}} /. eq // Simplify;
% // MatrixForm
```

$$\begin{pmatrix} \frac{2 \epsilon (\alpha+\gamma+\nu) (\alpha \lambda+\gamma (\lambda-\nu)+\lambda \nu)}{\beta (\alpha+\nu)} & -\frac{\epsilon (\lambda-\nu) \left(\alpha^2+3 \alpha \gamma+2 \gamma^2+2 \alpha \nu+3 \gamma \nu+\nu^2\right)}{\beta (\alpha+\nu)} \\ -\frac{\epsilon (\lambda-\nu) \left(\alpha^2+3 \alpha \gamma+2 \gamma^2+2 \alpha \nu+3 \gamma \nu+\nu^2\right)}{\beta (\alpha+\nu)} & \frac{2 \epsilon (\lambda-\nu) (\alpha+\gamma+\nu)^2}{\beta (\alpha+\nu)} \end{pmatrix}$$

```
Σ = {{Σ11, Σ12}, {Σ12, Σ22}};
A.Σ + Transpose[A.Σ] + B2 // Flatten
```

$$\left\{\frac{2 \epsilon (\alpha+\gamma+\nu) (\alpha \lambda+\gamma (\lambda-\nu)+\lambda \nu)}{\beta (\alpha+\nu)}+\frac{2 \gamma (-\lambda+\nu) \Sigma 11}{\alpha+\nu}+2 (-\alpha-\nu) \Sigma 12,\right.$$
$$-\frac{\epsilon (\lambda-\nu) \left(\alpha^2+3 \alpha \gamma+2 \gamma^2+2 \alpha \nu+3 \gamma \nu+\nu^2\right)}{\beta (\alpha+\nu)}+\frac{(\lambda-\nu) (\alpha+\gamma+\nu) \Sigma 11}{\alpha+\nu}+\frac{\gamma (-\lambda+\nu) \Sigma 12}{\alpha+\nu}+(-\alpha-\nu) \Sigma 22,$$
$$-\frac{\epsilon (\lambda-\nu) \left(\alpha^2+3 \alpha \gamma+2 \gamma^2+2 \alpha \nu+3 \gamma \nu+\nu^2\right)}{\beta (\alpha+\nu)}+\frac{(\lambda-\nu) (\alpha+\gamma+\nu) \Sigma 11}{\alpha+\nu}+\frac{\gamma (-\lambda+\nu) \Sigma 12}{\alpha+\nu}+(-\alpha-\nu) \Sigma 22,$$
$$\left.\frac{2 \epsilon (\lambda-\nu) (\alpha+\gamma+\nu)^2}{\beta (\alpha+\nu)}+\frac{2 (\lambda-\nu) (\alpha+\gamma+\nu) \Sigma 12}{\alpha+\nu}\right\}$$

```
Solve[
  {0 == 2 ϵ (α + γ + ν) (α λ + γ (λ - ν) + λ ν) / (β (α + ν)) + 2 γ (-λ + ν) Σ11 / (α + ν) + 2 (-α - ν) Σ12,
   0 == -ϵ (λ - ν) (α^2 + 3 α γ + 2 γ^2 + 2 α ν + 3 γ ν + ν^2) / (β (α + ν)) +
     (λ - ν) (α + γ + ν) Σ11 / (α + ν) + γ (-λ + ν) Σ12 / (α + ν) + (-α - ν) Σ22,
   0 == -ϵ (λ - ν) (α^2 + 3 α γ + 2 γ^2 + 2 α ν + 3 γ ν + ν^2) / (β (α + ν)) + (λ - ν) (α + γ + ν) Σ11 / (α + ν) +
     γ (-λ + ν) Σ12 / (α + ν) + (-α - ν) Σ22,
   0 == 2 ϵ (λ - ν) (α + γ + ν)^2 / (β (α + ν)) + 2 (λ - ν) (α + γ + ν) Σ12 / (α + ν)},
  {Σ11, Σ12, Σ12, Σ22}] // Flatten // Simplify;
Σ = Σ /. %;
% // MatrixForm
```

$$\begin{pmatrix} \frac{\epsilon (\alpha+\gamma+\nu) \left(\alpha^2+\gamma (\lambda-\nu)+\nu (\lambda+\nu)+\alpha (\lambda+2 \nu)\right)}{\beta \gamma (\lambda-\nu)} & -\frac{\epsilon (\alpha+\gamma+\nu)}{\beta} \\ -\frac{\epsilon (\alpha+\gamma+\nu)}{\beta} & \frac{\epsilon (\alpha+\gamma+\nu)^2 (\alpha+\lambda+\nu)}{\beta \gamma (\alpha+\nu)} \end{pmatrix}$$

```
Cov[τ_] := MatrixExp[A Abs[τ]].Σ;
```

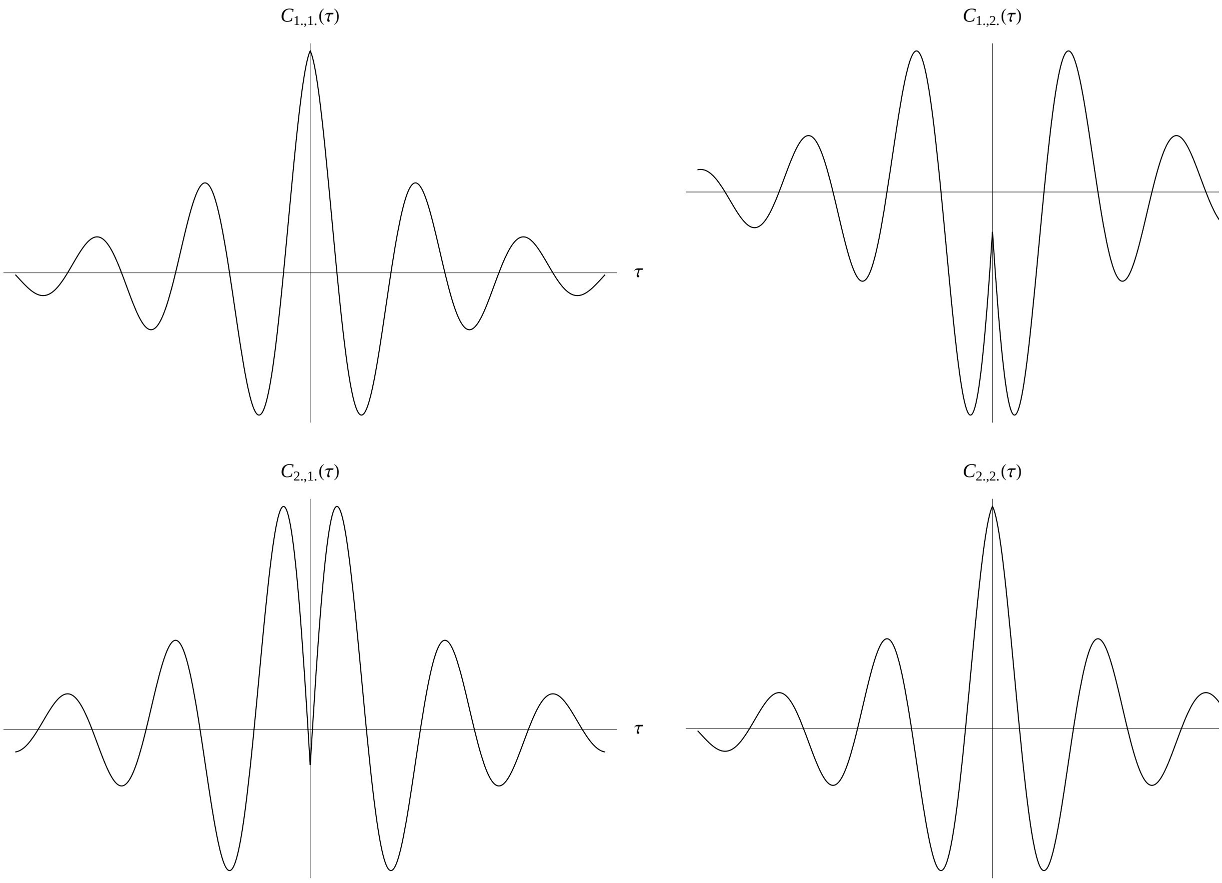
```
λ = 2; ν = 1; α = 1; β = 1; γ = 1; ϵ = .01;
```

```
Σ // MatrixForm
```

```
Grid[
 Table[If[i == 1.5 ∨ j == 1.5, , Plot[Cov[τ][[i, j]],
    {τ, -10, 10}, PlotStyle → Black, Ticks → None, AxesLabel → {τ, C_{i,j}[τ]}]],
  {i, 1, 2, .5}, {j, 1, 2, .5}]]
```

```
Clear[λ, ν, α, β, γ, ϵ, s, i];
```

$$\begin{pmatrix} 0.27 & -0.03 \\ -0.03 & 0.18 \end{pmatrix}$$

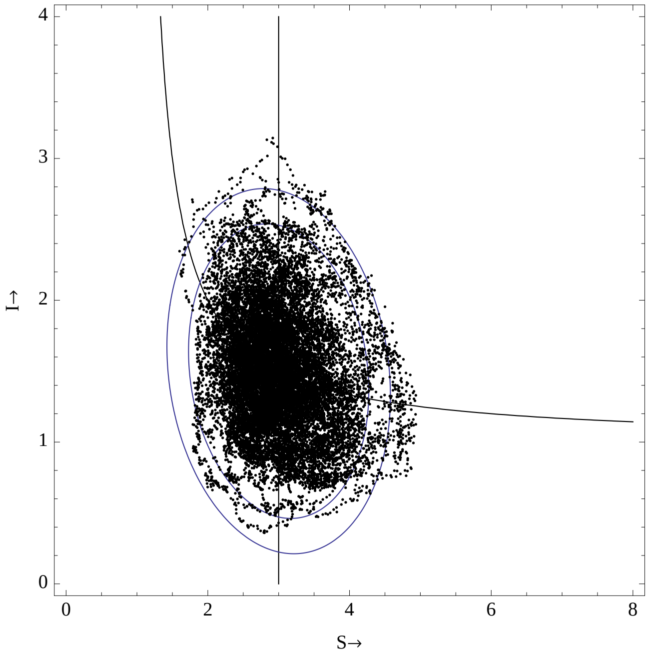


Stationary distribution OU approximation

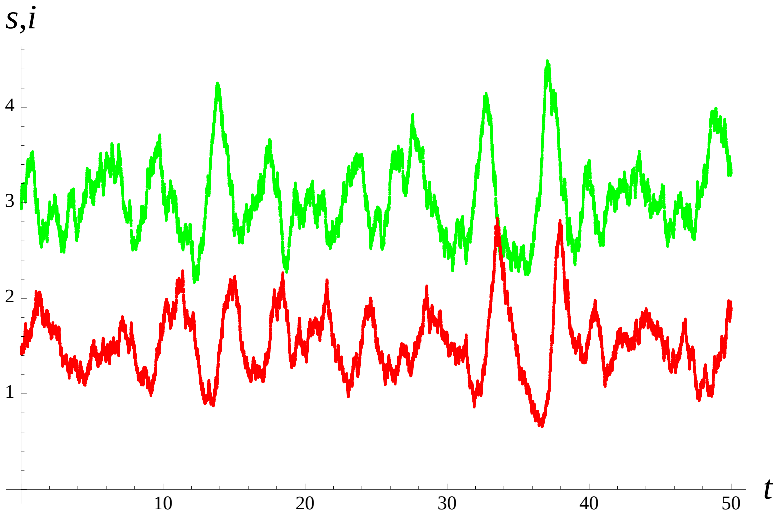
```
eq = {s, i} /. Solve[{μs == 0, μi == 0}, {s, i}] // Last
```

$$\left\{\frac{\alpha+\gamma+\nu}{\beta}, \frac{(\lambda-\nu)\,(\alpha+\gamma+\nu)}{\beta\,(\alpha+\nu)}\right\}$$

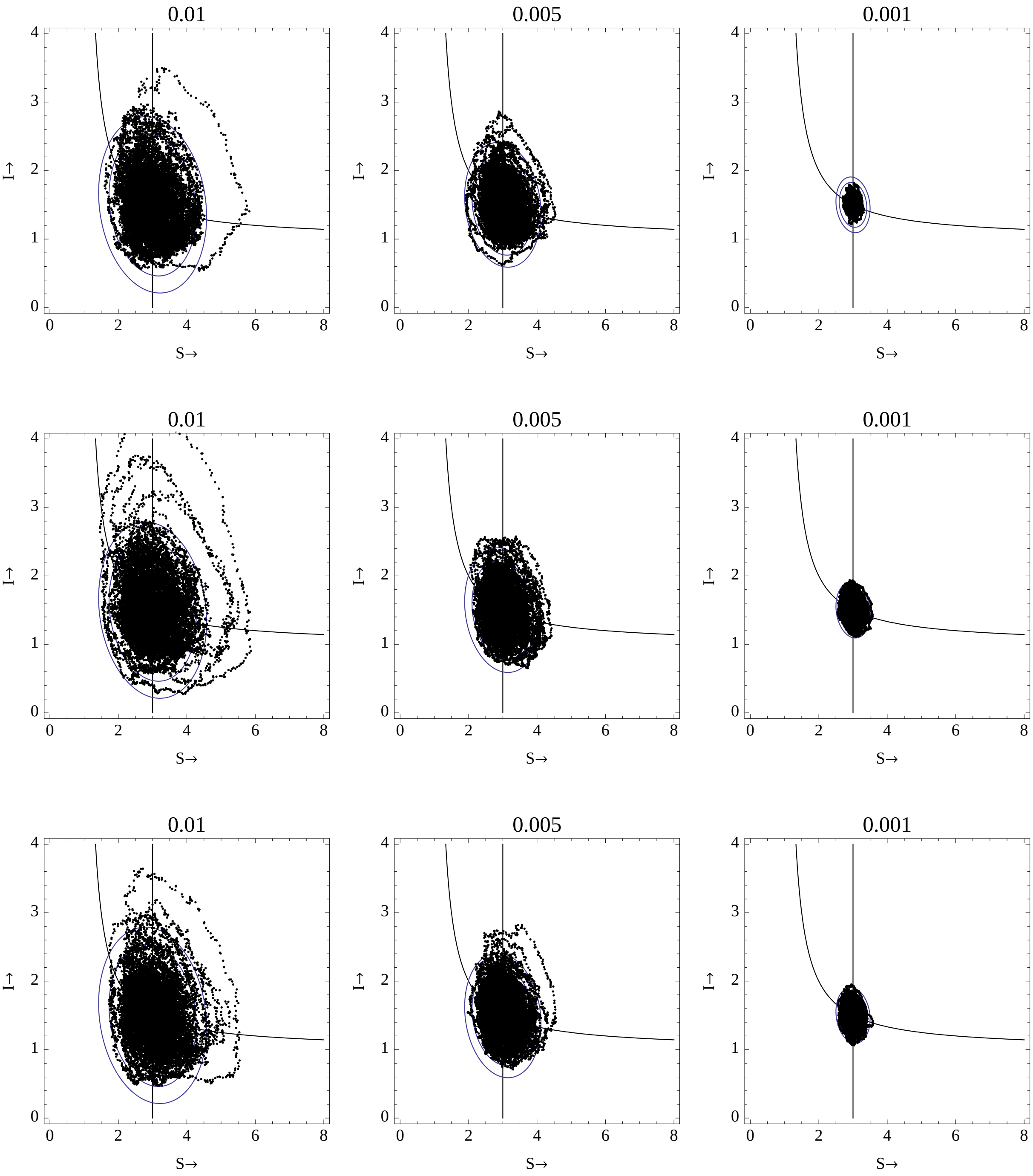
```
λ = 2; ν = 1; α = 1; β = 1; γ = 1; ϵ = .01;
dt = 0.01; tmax = 200;
{s, i} = eq; t = 0; dat = {};
While[t ≤ tmax,
  {ds, di} =
   {μs, μi} dt +
    RandomReal[MultinormalDistribution[{0, 0}, ϵ {{σss, σsi}, {σis, σii}} dt]];
  s = s + ds; i = i + di; t = t + dt;
  dat = Join[dat, {{s, i}}];
 ];
sdi = ListPlot[dat, Joined → False, PlotStyle → {Black, PointSize[.001]},
   PlotRange → All, MaxPlotPoints → Infinity, ImageSize → Small];
dist = MultinormalDistribution[eq, Σ];
q95 = Graphics[EllipsoidQuantile[dist, .95]];
q99 = Graphics[EllipsoidQuantile[dist, .99]];
Show[isp, q95, q99, sdi, FrameLabel → {"S→", "I→"}]
Clear[λ, ν, α, β, γ, ϵ, s, i];
```



```
λ = 2; ν = 1; α = 1; β = 1; γ = 1; ϵ = .01;
dt = 0.001; tmax = 50;
{s, i} = eq; t = 0; datS = {}; datI = {};
While[t ≤ tmax,
  {ds, di} =
   {μs, μi} dt +
    RandomReal[MultinormalDistribution[{0, 0}, ϵ {{σss, σsi}, {σis, σii}} dt]];
  s = s + ds; i = i + di; t = t + dt;
  datS = Join[datS, {{t, s}}];
  datI = Join[datI, {{t, i}}];
 ];
spS = ListPlot[datS, Joined → False,
   PlotStyle → {Green, PointSize[.001]}, PlotRange → All, MaxPlotPoints → Infinity];
spI = ListPlot[datI, Joined → False, PlotStyle → {Red, PointSize[.001]},
   PlotRange → All, MaxPlotPoints → Infinity];
Show[spS, spI, AxesOrigin → {0, 0}, ImageSize → Medium, AxesLabel → {"t", "s,i"}]
Clear[λ, ν, α, β, γ, ϵ, s, i];
```




```
λ = 2; ν = 1; α = 1; β = 1; γ = 1; ϵ = .001;
dt = 0.01; tmax = 200;
{s, i} = eq; t = 0; dat = {};
While[t ≤ tmax,
  {ds, di} =
   {μs, μi} dt +
    RandomReal[MultinormalDistribution[{0, 0}, ϵ {{σss, σsi}, {σis, σii}} dt]];
  s = s + ds; i = i + di; t = t + dt;
  dat = Join[dat, {{s, i}}];
 ];
sdi = ListPlot[dat, Joined → False,
   PlotStyle → {Black, PointSize[.001]}, PlotRange → All, MaxPlotPoints → Infinity];
dist = MultinormalDistribution[eq, Σ];
q50 = Graphics[EllipsoidQuantile[dist, .50]];
q95 = Graphics[EllipsoidQuantile[dist, .95]];
q99 = Graphics[EllipsoidQuantile[dist, .99]];
Show[isp, q50, q95, q99, sdi, FrameLabel → {"S→", "I→"}, PlotLabel → ϵ, ImageSize → Small]

Clear[λ, ν, α, β, γ, ϵ, s, i];
```


0.01
0.005
0.001
0.01
0.005
0.001
0.01
0.005
0.001
S→
I→
0
2
4
6
8
1
3

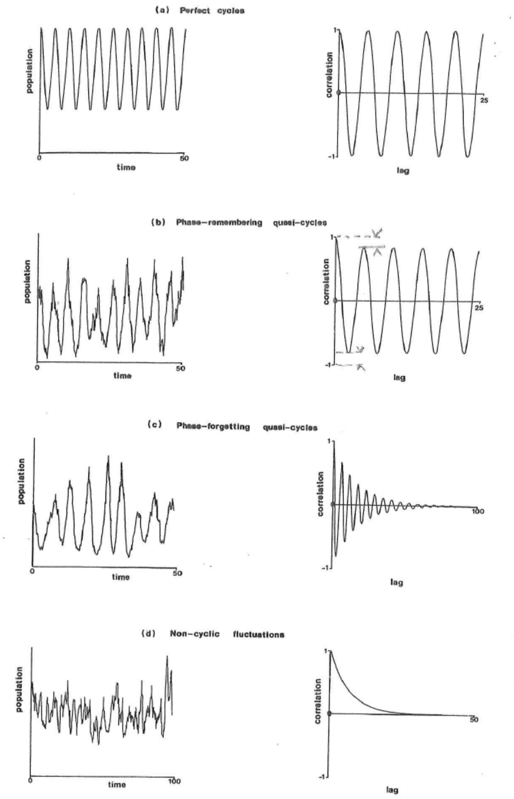


Fig. 1.6. Population fluctuations and their autocovariance functions. The left-hand plots show segments of four distinct population histories and the right-hand plots show the equivalent theoretical ACFs determined from the entire time history.

Copied from Nisbet & Gurney (1982)