Ito-calculus

22-02-2017

Infinitesimals (non-standard mathematics)

(short-hand for working with limits)

- Arbitrarily small but positive quantity. $dt > 0$.

- $\boxed{(dt)^p = 0}$ for $p > 1$.

- $dW(t) := W(t+dt) - W(t)$
(infinitesimal wiener increment)
$\sim \mathcal{N}(0, dt)$

- $dt\, dW(t) \sim \mathcal{N}(0, (dt)^3) = \mathcal{N}(0,0)$.
(Dirac-δ)
$\Rightarrow \boxed{dt\, dW(t) = 0}$ a.s.

- $(dW(t))^2 \sim$ Chi-square$_1$ with mean dt and var $2(dt)^2 = 0$
$\Rightarrow \boxed{(dW(t))^2 = dt}$ a.s.

- Ito's multiplication table:

	dt	$dW(t)$
dt	0	0
$dW(t)$	0	dt

Itô's Lemma.

Infinitesimal increments.

- $dW(t) := W(t+dt) - W(t)$
- $dX(t) := X(t+dt) - X(t)$

where $X(t)$ is ordinary func.

- $Y(t) := h(X(t))$

$$
\begin{aligned}
dY(t) &:= Y(t+dt) - Y(t) \\
&= h(X(t+dt)) - h(X(t)) \\
&= h(X(t) + dX(t)) - h(X(t)) \\
&= h'(X(t))\,dX(t) + \\
&\quad + \tfrac{1}{2}h''(X(t))\,(dX(t))^2 + O(dX(t))^3
\end{aligned}
$$

- If X were an ordinary function of time, then

$$
\begin{aligned}
dX(t) &= X(t+dt) - X(t) \\
&= x'(t)\,dt + \cancel{O(dt^2)}
\end{aligned}
$$

and so

$$(dX(t))^2 = (x'(t))^2(dt)^2 = 0$$

and

$$\boxed{dY = h'(X)\,dX}$$

i.e.

$$\boxed{\frac{dY}{dt} = h'(X)\frac{dX}{dt}}$$

normal chain rule

- If $X(t) = W(t)$ (Wiener process)

then

$$(dX(t))^2 = (dW(t))^2 = dt$$

(and $(dW(t))^3 = (dt)^{3/2} = 0$)

and so

$$\boxed{dY = h'(W)\,dW + \tfrac{1}{2}h''(W)\,dt}$$

i.e.

$$\boxed{\frac{dY}{dt} = \tfrac{1}{2}h''(W) + h'(W)\frac{dW}{dt}}$$

← noise induced drift

~~In other words, if Y satisfies the SDE ⊛, then~~

$$\cancel{Y(t) = h(W(t)) + h(W(0))}$$

~~is the solution.~~

$W(0)=0$ a.s.

This is essentially Ito's Lemma. Gives you the chain rule of differentiation of functions of noise:

$$\boxed{\frac{dh(W)}{dt} = \underbrace{h'(W)\frac{dW}{dt}}_{\text{usual}} + \underbrace{\tfrac{1}{2}h''(W)}_{\text{new}}}$$

Ito calculus is all about the chain rule ⊛

It is all about the chain rule

Example

$$Y = h(W) := W^2.$$

What is $\int_0^t W(s)\,dW(s) = \int_0^t W(s)\,\xi(s)\,ds$?

$$h'(W) = 2W$$

$$h''(W) = 2.$$

white noise

diff $\Rightarrow$ chain rule

$$\frac{dW(t)^2}{dt} = 1 + 2W(t)\frac{dW(t)}{dt}$$

Take $W(0) = 0$ a.s. and integrate over $[0, t]$.

integr $\Rightarrow$

$$W(t)^2 = t + 2\int_0^t W(s)\,dW(s)$$

Hence, we find that the stochastic integral

$$\boxed{\int_0^t W(s)\,dW(s) = \frac{1}{2}W(t)^2 - \frac{t}{2}}$$

Now we know how to evaluate this integral.

$W(t)^2 \sim \chi^2_1$ with mean t and variance $2t^2$

$\Rightarrow \int_0^t W(s)\,dW(s) \sim \chi$ drift

Example

$$Y(t) = e^{\sigma W(t)}$$ ~ logNormal.

$$dY = \tfrac{1}{2}\sigma^2 Y\,dt + \sigma Y\,dW$$

Example

$$Y(t) = e^{-\frac{1}{2}\sigma^2 t + \sigma W(t)}.$$

~ logNormal.

$$dY = \sigma Y\,dW$$

(stoch. analog of the ODE $dy = \sigma y\,dt$.)

Example

don't remember the formulas, but remember Itô's multiplication table.

• $\boxed{dX = f(X)\,dt + g(X)\,dW}$. (SDE)

$\boxed{Y := h(X)}$

$$dY(t) = Y(t+dt) - Y(t)$$

$$= h(X(t+dt)) - h(X(t))$$

$$= h(X+dX) - h(X)$$

$$= h'(X)\,dX + \tfrac{1}{2} h''(X)(dX)^2 + O(dX)^3$$

$$= h'(X)\big(f(X)\,dt + g(X)\,dW\big) + \tfrac{1}{2} h''(X)\big(\text{———}\ \text{''}\ \text{———}\big)^2 + O(dX)^3$$

$$= \Big(h'(X)f(X) + \tfrac{1}{2} h''(X)\, g(X)^2\Big)\,dt + h'(X)\, g(X)\, dW$$

$$\boxed{\frac{dh}{dt} = h'f + \tfrac{1}{2} h'' g^2 + h' g \frac{dW}{dt}}$$

(chain rule in Ito calculus)