Interference competition and noise

Model 1

Individual model :

$$X \xrightarrow{\beta[t]} X + X \quad (\textit{birth})$$

$$X + X \xrightarrow{2\gamma} X + \emptyset \quad (\textit{interference})$$

$$X \xrightarrow{\delta} \emptyset \quad (\textit{death})$$

modelled as independent Poisson processes and assuming mass - action when appropriate.

Environmental noise :

$$\beta[t] := \beta_0\, e^{\theta[t]}$$

$$d\theta == -a\,\theta\, dt + b\, dW$$

with $a > 0$.

Population model :

$$\begin{cases} dX == \left(\beta_0\, e^{\theta}\, X - \delta\, X - \gamma\, X^2\right) dt \\ d\theta == -a\,\theta\, dt + b\, dW \end{cases}$$

with a > 0.

Linear approximation :

$$\overline{X} = (\beta_0 - \delta)\ /\ \gamma$$

$$U = X - \overline{X}$$

$$\begin{cases} dU == -(\beta_0 - \delta)\, U\, dt + \beta_0\, (\beta_0 - \delta)\, \gamma^{-1}\, \theta\, dt \\ d\theta == -a\,\theta\, dt + b\, dW \end{cases}$$

with $\beta_0 > \delta$ and a > 0.

Ornstein - Ulenbeck process :

$U \sim N\left[0, \sigma^2\right]$ **and hence** $X \sim N\left[\overline{X}, \sigma^2\right]$ **with** $\overline{X} = (\beta_0 - \delta)\, \gamma^{-1}$

$$S_1[\omega] = \frac{(\beta_0 - \delta)^2\, \beta_0^2}{\gamma^2\left(\omega^2 + (\beta_0 - \delta)^2\right)}\ \frac{b^2}{\omega^2 + a^2}$$

$$C_1[t] = \begin{cases} \dfrac{b^2\, \beta_0^2\, (\beta_0 - \delta)\, e^{-t\,(a + \beta_0 - \delta)} \left(a\, e^{t\,a} - (\beta_0 - \delta)\, e^{t\,(\beta_0 - \delta)}\right)}{2\, a\, \gamma^2 \left(a^2 - (\beta_0 - \delta)^2\right)} & \text{if } \beta_0 \neq a + \delta \\ \dfrac{b^2\, e^{-a\,t}\, (1 + a\,t)\, (a + \delta)^2}{4\, a\, \gamma^2} & \text{if } \beta_0 = a + \delta \end{cases}$$

for t > 0, and so, in particular,

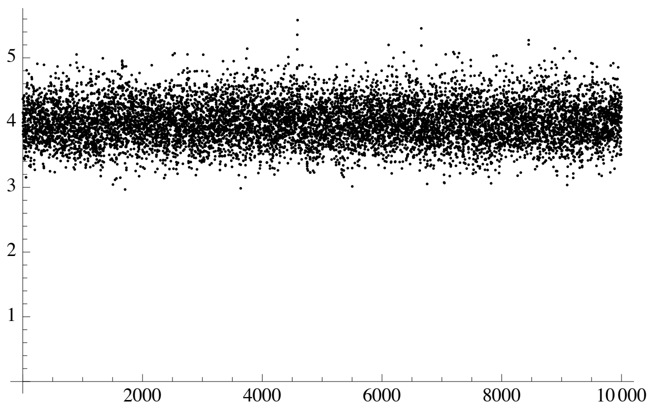
$$\sigma_1^2 = \begin{cases} \dfrac{b^2\, \beta_0^2\, (\beta_0 - \delta)}{2\, a\, \gamma^2\, (a + \beta_0 - \delta)} & \text{if } \beta_0 \neq a + \delta \\ \dfrac{b^2\, (a + \delta)^2}{4\, a\, \gamma^2} & \text{if } \beta_0 = a + \delta \end{cases}$$

```
In[1]:= << MultivariateStatistics`;

β0 = 5; γ = 1; δ = 1; a = 1; b = .1;
(* β0=2; *)
SeedRandom[1234];
tMax = 10 000;
dt = .01;
t = 0;
x = (β0 - δ) / γ;
θ = 0;
data1 = {};
While[
  t < tMax,
  Do[
   z = RandomReal[NormalDistribution[0, 1]] // N;
   dθ = -a θ dt + Sqrt[dt] b z;
   dx = (β0 ⅇ^θ x - δ x - γ x^2) dt;
   t = t + dt;
   θ = θ + dθ;
   x = x + dx,
   {i, 1, 100}];
  data1 = Join[data1, {{t, x}}]];

In[11]:= ListPlot[data1, PlotStyle → {Black, PointSize[.001]}, AxesOrigin → {0, 0}]
```

Out[11]=

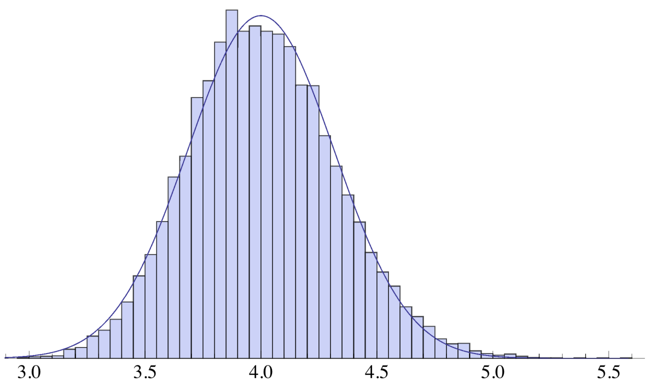


In[12]:=
```
μ = (β0 - δ) / γ;
```

$$\sigma = \sqrt{\text{If}\left[\beta_0 \neq a + \delta,\ \frac{b^2\,\beta_0^2\,(\beta_0 - \delta)}{2\,a\,\gamma^2\,(a + \beta_0 - \delta)},\ \frac{b^2\,(a + \delta)^2}{4\,a\,\gamma^2}\right]}\ ;$$

```
distr1 =
 Show[
  Histogram[data1[[All, 2]], Automatic, "ProbabilityDensity"],
  Plot[PDF[NormalDistribution[μ, σ], x], {x, 0, Max[data1[[All, 2]]]}, PlotRange → All],
  AxesOrigin → {0, 0}
 ]
```

Out[14]=



Model 2

Individual model :

$$X \xrightarrow{\beta[t]} X + X \quad (\textbf{birth})$$
$$X + X \xrightarrow{2\gamma} X + \emptyset \quad (\textbf{interference})$$
$$X \xrightarrow{\delta} \emptyset \quad (\textbf{death})$$

modelled as independent Poisson processes and assuming mass - action when appropriate.

Environmental noise :

$$\beta[t] := \beta_0 + \epsilon\, \xi[t]$$

where $\xi[t]$ ***is the standard white noise.***

Population model :

$$dX == \left(\beta_0 X - \delta X - \gamma X^2\right) dt + \epsilon X\, dW$$

Linear approximation :

$$\overline{X} = \frac{(\beta_0 - \delta)}{\gamma}$$

$$U = X - \overline{X}$$

$$dU == -(\beta_0 - \delta)\, U\, dt + \epsilon \frac{(\beta_0 - \delta)}{\gamma} dW$$

Ornstein - Ulenbeck process :

$U \sim N\left[0, \sigma^2\right]$ **and hence** $X \sim N\left[\overline{X}, \sigma^2\right]$ **with** $\overline{X} = (\beta_0 - \delta)\ \gamma^{-1}$

$$S_2[\omega] = \frac{\epsilon^2\ (\delta - \beta_0)^2}{\gamma^2\ \left(\omega^2 + (\beta_0 - \delta)^2\right)}$$

$$2[t] = \frac{\epsilon^2\ (\beta_0 - \delta)}{2\ \gamma^2}\ e^{-(\beta_0 - \delta)\ t}$$

for $t > 0$, ***and in particular***

$$\sigma_2^2 = \frac{\epsilon^2\ (\beta_0 - \delta)}{2\ \gamma^2}$$

Since we have no natural interpretation of the ϵ***,***
we have to callibrate the variances of the two models
such that they become equal, which gives

$$\epsilon^2 = \mathrm{If}\left[\beta_0 \neq a + \delta,\ \frac{b^2\ \beta_0^2}{a\ (a + \beta_0 - \delta)},\ \frac{b^2\ (a + \delta)^2}{2\ a\ (\beta_0 - \delta)}\right]$$

In[15]:=
```
<< MultivariateStatistics`;

β0 = 5; γ = 1; δ = 1; a = 1; b = .1;
(* β0=2; *)
```

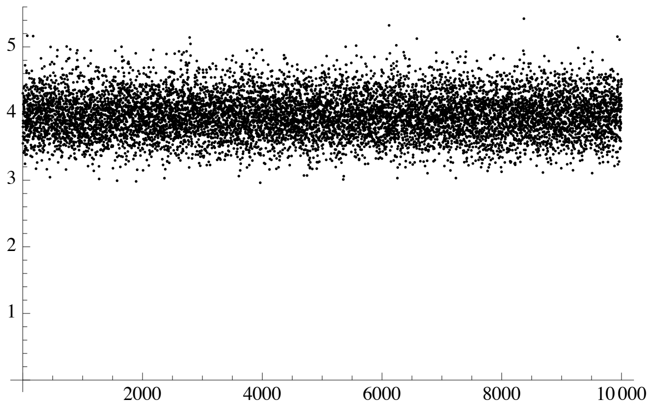
$$\epsilon = \sqrt{\text{If}\left[\beta_0 \neq a + \delta,\ \frac{b^2\,\beta_0^2}{a\,(a + \beta_0 - \delta)},\ \frac{b^2\,(a + \delta)^2}{2\,a\,(\beta_0 - \delta)}\right]};$$

```
(* this ϵ equalizes the vars of the two models *)

SeedRandom[1234];
tMax = 10 000;
dt = .01;
t = 0;
x = (β0 - δ) / γ;
θ = 0;
data2 = {};
While[
  t < tMax,
  Do[
   z = RandomReal[NormalDistribution[0, 1]] // N;
   dx = (β0 x - δ x - γ x^2) dt + ϵ x Sqrt[dt] z;
   t = t + dt;
   x = x + dx,
   {i, 1, 100}];
  data2 = Join[data2, {{t, x}}]];
```

In[26]:=
```
ListPlot[data2, PlotStyle → {Black, PointSize[.001]}, AxesOrigin → {0, 0}]
```

Out[26]=



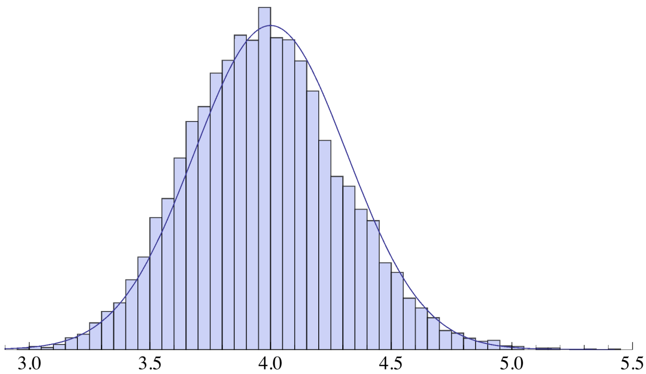
In[27]:=

```
μ = (β₀ - δ) / γ;
```

$$\sigma = \sqrt{\frac{\epsilon^2\ (\beta_0 - \delta)}{2\,\gamma^2}};$$

```
distr₂ =
 Show[
  Histogram[data₂[[All, 2]], Automatic, "ProbabilityDensity"],
  Plot[PDF[NormalDistribution[μ, σ], x], {x, 0, Max[data₂[[All, 2]]]}, PlotRange → All],
  AxesOrigin → {0, 0}
 ]
```

Out[29]=



Comparing models 1 and 2

The difference between the models is not in the stationary distribution,
but in the time structure

In[30]:=
```
β0 = 5; γ = 1; δ = 1; a = 1; b = .1;
(* β0=2; *)
```

$$\epsilon = \sqrt{\text{If}\left[\beta_0 \neq a + \delta,\ \frac{b^2\,\beta_0^2}{a\,(a+\beta_0-\delta)},\ \frac{b^2\,(a+\delta)^2}{2\,a\,(\beta_0-\delta)}\right]};$$

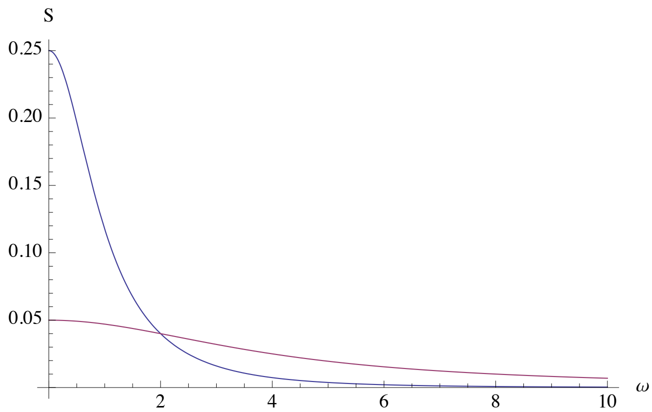
In[32]:=

$$S_1[\omega\_] := \frac{b^2\,(\delta-\beta_0)^2\,\beta_0^2}{\gamma^2\,\left(a^2+\omega^2\right)\,\left(\omega^2+(\delta-\beta_0)^2\right)};$$

$$S_2[\omega\_] := \frac{\epsilon^2\,(\delta-\beta_0)^2}{\gamma^2\,\left(\omega^2+(\beta_0-\delta)^2\right)};$$

```
Plot[{S1[ω], S2[ω]}, {ω, 0, 10}, PlotRange → All, AxesLabel → {"ω", "S"}]
```

Out[34]=



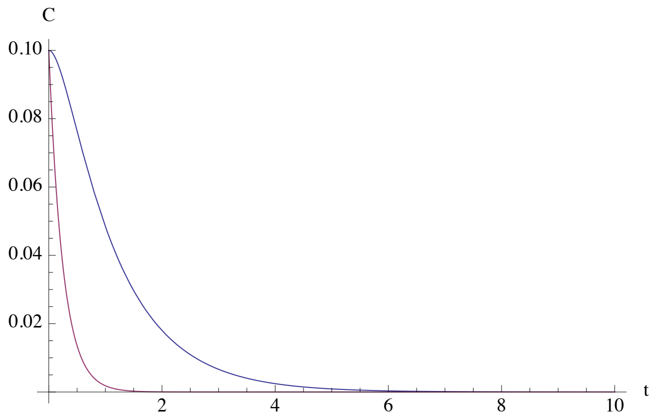
In[35]:=

$$\mathbf{Cov_1[t\_]} := \mathbf{If}\left[\beta_0 \neq a + \delta,\ \frac{b^2\,\beta_0^2\,(\beta_0 - \delta)\,e^{-t\,(a+\beta_0-\delta)}\left(a\,e^{t\,a} - (\beta_0 - \delta)\,e^{t\,(\beta_0-\delta)}\right)}{2\,a\,\gamma^2\left(a^2 - (\beta_0 - \delta)^2\right)},\ \frac{b^2\,e^{-a\,t}\,(1 + a\,t)\,(a + \delta)^2}{4\,a\,\gamma^2}\right];$$

$$\mathbf{Cov_2[t\_]} := \frac{\epsilon^2\,(\beta_0 - \delta)}{2\,\gamma^2}\,e^{-(\beta_0-\delta)\,t};$$

```
Plot[{Cov1[t], Cov2[t]}, {t, 0, 10}, PlotRange → All, AxesLabel → {"t", "C"}]
```

Out[37]=

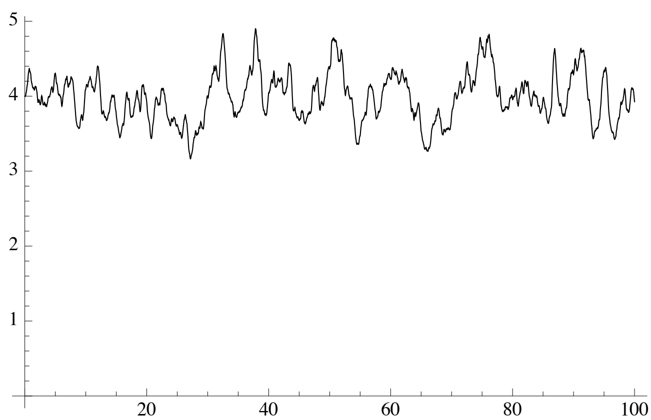


In[38]:=

```
SeedRandom[1234];
tMax = 100;
dt = .005;
t = 0;
x = (β0 - δ) / γ;
θ = 0;
data1 = {};
While[
  t < tMax,
  Do[
   z = RandomReal[NormalDistribution[0, 1]] // N;
   dθ = -a θ dt + √dt b z;
   dx = (β0 e^θ x - δ x - γ x^2) dt;
   t = t + dt;
   θ = θ + dθ;
   x = x + dx,
   {i, 1, 1}]; (* collect full data *)
  data1 = Join[data1, {{t, x}}]];

ListPlot[data1, PlotStyle → {Black}, Joined → True, AxesOrigin → {0, 0}]
```

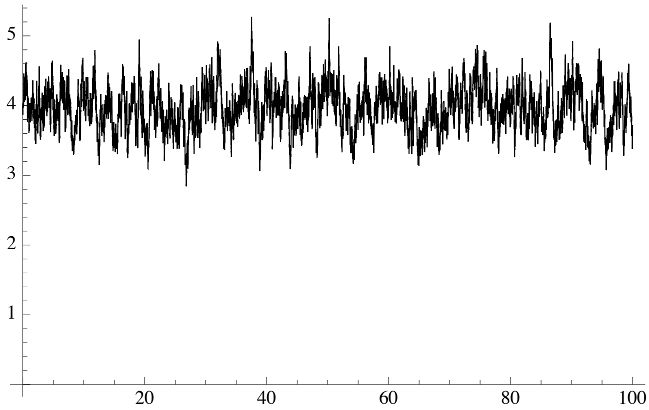
Out[46]=



In[47]:=
```
SeedRandom[1234];
tMax = 100;
dt = .005;
t = 0;
x = (β0 - δ) / γ;
θ = 0;
data2 = {};
While[
  t < tMax,
  Do[
   z = RandomReal[NormalDistribution[0, 1]] // N;
   dx = (β0 x - δ x - γ x^2) dt + ϵ x Sqrt[dt] z;
   t = t + dt;
   x = x + dx,
   {i, 1, 1}]; (* collect full data *)
  data2 = Join[data2, {{t, x}}]];

ListPlot[data2, PlotStyle → {Black}, Joined → True, AxesOrigin → {0, 0}]
```

Out[55]=



In[56]:=
```
Row[{
  Show[
   DensityPlot[PDF[NormalDistribution[μ, σ], x], {t, 0, tMax}, {x, μ - 4 σ, μ + 4 σ}],
   ListPlot[data1, PlotStyle → {Black}, Joined → True], ImageSize → Small],
  Show[
   DensityPlot[PDF[NormalDistribution[μ, σ], x], {t, 0, tMax}, {x, μ - 4 σ, μ + 4 σ}],
   ListPlot[data2, PlotStyle → {Black}, Joined → True], ImageSize → Small]
 }]
```

Out[56]=