STOCHASTIC POPULATION MODELS

EXERCISES 5-8

5.
Consider the epidemiological model of the last exercise based on the processes

$$\begin{aligned} \mathrm{S} + \mathrm{I} &\xrightarrow{\alpha} 2\mathrm{I} \quad \text{(transmission)} \\ \mathrm{I} &\xrightarrow{\beta} \mathrm{S} \quad \text{(recovery)} \end{aligned}$$

Give the transfer function $T(\omega)$ for a seasonally varying infection rate α. Calculate the maximum gain G_{m} and the cutoff frequency ω_{c}. How does the cutoff frequency depend on the average infection rate? Express the cutoff frequency in units of *cycles* per *mean recovery time* (β^{-1}). What does this tell you about absence (or presence) of seasonality of the prevalence of the disease in relation to the average length of an infection?

6.
Consider the resource-consumer model

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}R = R_0 - \alpha R - \beta R C & \text{(resource)} \\ \frac{\mathrm{d}}{\mathrm{d}t}C = \gamma\beta R C - \alpha C - \delta C^2 & \text{(consumer)} \end{cases}$$

Can you interpret the different terms and parameters? Calculate the equilibria and determine their stability. For the positive equilibrium, give the transfer functions $T_{\mathrm{R}}(\omega)$ and $T_{\mathrm{C}}(\omega)$ for, respectively, the resource and the consumer if the R_0 is seasonally varying. Under what conditions is there a resonance peak?

7.
Let $\tilde{f}$ and $\tilde{h}$ denote the Fourier transforms of, respectively, f and h, and show that:

(a) the Fourier transform and its inverse are linear operators,

(b) $\tilde{\tilde{f}}(t) = 2\pi f(-t)$,

(c) $\widetilde{(\frac{d}{dt}f)}(\omega) = i\omega\tilde{f}(\omega)$,

(d) $\frac{d}{d\omega}\tilde{f}(\omega) = -i\widetilde{(tf)}(\omega)$,

(e) $\tilde{f}_\tau(\omega) = e^{-i\omega\tau}\tilde{f}(\omega)$ where $f_\tau(t) := f(t-\tau)$,

(f) $\widetilde{(f*h)}(\omega) = \tilde{f}(\omega)\tilde{h}(\omega)$ where $(f*h)(t) := \int_{-\infty}^{+\infty} f(\tau)h(t-\tau)d\tau$,

(g) $\widetilde{(fh)}(\omega) = \frac{1}{2\pi}(\tilde{f}*\tilde{h})(-\omega)$,

(h) $\int_{-\infty}^{+\infty} f(t)\tilde{h}(t)dt = \int_{-\infty}^{+\infty} \tilde{f}(t)h(t)dt$.

8.
Let δ denote the Dirac-delta distribution, and show that:

(a) $\widetilde{(t^n)}(\omega) = 2\pi\, n!\, \delta(\omega)/(i\omega)^n$ for $n = 0, 1, \ldots$

(b) $\widetilde{e^{i\omega_0 t}}(\omega) = 2\pi\, \delta(\omega - \omega_0)$

(c) $\widetilde{\cos(\omega_0 t)}(\omega) = \pi\, \delta(\omega - \omega_0) + \pi\, \delta(\omega + \omega_0)$