STOCHASTIC POPULATION MODELS

EXERCISES 13-14

13.
Consider the stochastic process $\{X(t)\}_{t\geq 0}$ generated by the linear stochastic differential equation

$$\mathrm{d}X = (a - bX)\mathrm{d}t + 2\sqrt{c}\,\mathrm{d}W \quad \text{(Ito)}$$

with $a, b, c > 0$ and initial condition $X(0) = x_0$ (a.s.).

(a) Give an ODE for the mean $\overline{X}(t) := \mathcal{E}\{X(t)\}$ and solve it. Under what conditions on a, b, c does $\overline{X}(t)$ converge as $t \to \infty$? What is this limit?

(b) Use Ito's multiplication table to get a SDE for X^n in terms of X^n itself and X^{n-1}. Next, get an ODE for the n-th moment $\mu_n(t) := \mathcal{E}\{X(t)^n\}$ in terms of μ_n itself and μ_{n-1}. Solve the equations recursively starting with $n = 1$, and check under what conditions on a, b, c and n the solution converges as $t \to \infty$.

(c) Integrate the SDE to get find the distribution of $X(t)$ for $t \geq 0$. Under what conditions on a, b, c does the SDE permit a stationary distribution? What does this distribution look like? Calculate the spectral density and auto-covariance for the stationary process.

(e) Give the Fokker-Plack equation corresponding to the SDE, and check that the probability distribution found in **(b)** satisfies the equation. Also, calculate the stationary distribution directly from the Fokker-Planck equation.

14.
Consider the stochastic process $\{X(t)\}_{t\geq 0}$ generated by the non-linear stochastic differential equation

$$\mathrm{d}X = (a - bX)\mathrm{d}t + 2\sqrt{c}\,X\,\mathrm{d}W \quad \text{(Ito)}$$

with $a, b, c > 0$ and initial condition $X(0) = x_0$ (a.s.).

(a) Give an ODE for the mean $\overline{X}(t) := \mathcal{E}\{X(t)\}$ and solve it. Under what conditions on a, b, c does $\overline{X}(t)$ converge as $t \to \infty$? What is this limit?

(b) Use Ito's multiplication table to get a SDE for X^n in terms of X^n itself and X^{n-1}. Next, get an ODE for the n-th moment $\mu_n(t) := \mathcal{E}\{X(t)^n\}$ in terms of μ_n

itself and μ_{n-1}. Solve the equations recursively starting with $n = 1$, and check under what conditions on a, b, c and n the solution converges as $t \to \infty$.

(c) Give the Fokker-Plack equation corresponding to the SDE, and find a stationary probability distribution directly from the Fokker-Planck equation.