STOCHASTIC POPULATION MODELS

EXERCISES 1-4

1. Model the following processes (separated by "&") as networks of monomolecular or bimolecular reactions, i.e., decide what kind of "molecules" you need and what "reactions" describe the processes best. For each network also give the corresponding population equations (ODEs) for the rate of change in the population densities using the principle of mass action.

(a) binary fission of a cell & cell death

(b) asexual reproduction & maturation of the offspring

(c) marriage & divorce

(d) transmission of a disease from an infected to an uninfected individual & recovery from the disease

(e) predator attacks and prey is captured & predator attacks but prey escapes

(f) predator detects prey and stealthily approaches & prey notices approaching predator and escapes & prey does not notice predator and is captured

(g) the meeting of an acquaintance in a crowded street & breaking up of the meeting after a handshake and a little chat

2. What is the difference between the following two modes of reproduction:

$$\mathrm{A} \xrightarrow{\alpha} 2\mathrm{A} \quad \text{and} \quad 2\mathrm{A} \xrightarrow{\alpha} 3\mathrm{A}$$

Give the corresponding population equations and solve explicitly using the same initial density of A. Plot the solution of each as a function of time in the same graph to see the difference in dynamics.

3. Let I denote an infected individual and S an uninfected but susceptible individual, and consider the following processes:

$$\begin{array}{rcll} \mathrm{S} + \mathrm{I} & \xrightarrow{\alpha} & 2\mathrm{I} & \text{(transmission)} \\ \mathrm{I} & \xrightarrow{\beta} & \mathrm{S} & \text{(recovery)} \end{array}$$

Use this model to derive a mechanistic underpinning for the logistic equation

$$\frac{dx}{dt} = rx(1 - \frac{x}{K})$$

where r and K are functions of the rates α and β. Make a graph of K versus r to show how K and r co-vary as a consequence of varying α and, separately, β. What is the range within α and β can meaningfully vary?

4. To get the time-scale separation between the dynamics of p and s in

$$\left\{ \begin{array}{rcrrr} \frac{\mathrm{d}p}{\mathrm{d}t} & = & & +\beta s(e_0 - p) & -\gamma p \\ \frac{\mathrm{d}s}{\mathrm{d}t} & = & \alpha p & -\beta s(e_0 - p) & -\delta s \end{array} \right.$$

(i.e, equation (20) in section 1.9 of the lecture notes) in a more rigorous way, let $\varepsilon > 0$ be a dimensionless scaling parameters, and write $\alpha = \alpha^* \varepsilon^{-1}$ and $\delta = \delta^* \varepsilon^{-1}$, so that reproduction and seed death become arbitrary fast as ε approaches zero.

Let further $t^* = t\, \varepsilon^{-1}$ denote the so-called *fast time* (as opposed to merely t which is referred to as *normal* or *slow time*). Derive the fast dynamics by rewriting the system for fast time, i.e., give equations for

$$\frac{\mathrm{d}p}{\mathrm{d}t^*} = \frac{\mathrm{d}p}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t^*} \quad \text{and} \quad \frac{\mathrm{d}s}{\mathrm{d}t^*} = \frac{\mathrm{d}s}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}t^*}$$

and then take the limit for $\varepsilon \to 0$. Solve the resulting dynamics.

Next, consider the original system in normal time, but multiply the equation for s with ε (both sides!) and take the limit for $\varepsilon \to 0$. Compare the resulting equations with equation (22) of the lecture notes. Why is it necessary to analyse the fast dynamics at all and not just look at the slow dynamics?