STOCHASTIC POPULATION MODELS

EXERCISES $9\frac{1}{2}$

$9\frac{1}{2}$.
In the lecture notes we prove a simple version of the Central Limit Theorem, but there are some gaps, which you are asked to fill. For reference the proof from the lecture notes has been copied below, but first the gaps:

(a) Show in (5) that $\tilde{p}_{\frac{Y}{\sqrt{n}}}(\omega) = \tilde{p}_Y\left(\frac{\omega}{\sqrt{n}}\right)$.

(b) In the paragraph after (5) show that the probability density of the sum of independent random variables is the convolution of the respective probability densities.

That is all for today.

Proposition. If $X_1, X_2, X_3, \dots$ are independent and identically distributed ramdom variables with finite mean μ and finite variance $\sigma^2 > 0$, then

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma\sqrt{n}} \sim \mathcal{N}(0,1) \tag{1}$$

Proof. We prove the Central Limit Theorem as follows, but there are some gaps which you are asked to fill. First define

$$Z_n := \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma\sqrt{n}} \tag{2}$$

and show that the probability density of Z_n converges to the the probability density of the standard normal distribution as $n \to \infty$, i.e., we prove *convergence in distribution*. It is easier, however, to show that the characteristic function of Z_n converges to the characteristic function of the standard normal distribution, which is the same thing.

To calculate the characteristic function of Z_n, first define

$$Y_i := \frac{X_i - \mu}{\sigma} \quad (i = 1, 2, \dots) \tag{3}$$

The Y_i then are independently and identically distributed random variables with zero mean and unit variance and with a probability density $p_Y(y)$ (which can be calculated from the density of the X_i, but we will not need that). The characteristic function of the Y_i is

$$
\begin{aligned}
\tilde{p}_Y(\omega) &= \int_{-\infty}^{+\infty} p_Y(y)\, e^{-i\omega y} dy \\
&= \int_{-\infty}^{+\infty} p_Y(y)\big[1 - i\omega y - \tfrac{1}{2}\omega^2 y^2 + \mathrm{O}(\omega^3)\big] dy \\
&= 1 - i\omega\mathcal{E}\{Y\} - \tfrac{1}{2}\omega^2\mathcal{E}\{Y^2\} + \mathrm{O}(\omega^3) \\
&= 1 - \tfrac{1}{2}\omega^2 + \mathrm{O}(\omega^3)
\end{aligned} \tag{4}
$$

Consequently, the characteristic function of the $Y_i/\sqrt{n}$ is

$$
\tilde{p}_{\frac{Y}{\sqrt{n}}}(\omega) = \tilde{p}_Y\left(\frac{\omega}{\sqrt{n}}\right) = 1 - \frac{\omega^2}{2n} + \mathrm{O}\left(\frac{\omega^3}{n^{3/2}}\right) \tag{5}
$$

Since the probability density of the sum of independent random variables is the convolution of the respective probability densities, the characteristic function of the sum of independent random variables is product of the respective characteristic functions. Applying this to

$$
Z_n = \sum_{i=1}^{n} \frac{Y_i}{\sqrt{n}} \tag{6}
$$

we find that the characteristic function of Z_n is

$$
\tilde{p}_{Z_n}(\omega) = \left(1 - \frac{\omega^2}{2n} + \mathrm{O}\left(\frac{\omega^3}{n^{3/2}}\right)\right)^n \tag{7}
$$

and hence

$$
\lim_{n\to\infty} \tilde{p}_{Z_n}(\omega) = e^{-\frac{1}{2}\omega^2} \tag{8}
$$

which is the characteristic function of the standard normal distribution (see Section 5.1). So, the characteristic function of Z_n converges to that of the standard normal distribution, which completes the proof of the Central Limit Theorem.