# ON THE PROBABILITY OF INVASION IN A MULTI-TYPE BRANCHING PROCESS WITH A SINGLE BIRTH STATE

STEFAN A. H. GERITZ

Consider a multi-type branching process with states $0, \dots, n$, and where 0 corresponds to the unique birth state, and let $b_j$ denote the birth rate and $d_j$ the death rate in state $j$, and let $t_{ij}$ be the transition rate from state $j$ to state $i$. For the conservation of probability mass we necessarily have

$$t_{jj} = -\sum_{i \neq j} t_{ij} \quad \forall j. \tag{1}$$

Let further $p_j[l]$ denote the probability that an individual presently in state $j$ will produce $l$ offspring during the rest of its stay in the same state $j$, and let $q_j(k)$ denote the probability that an individual presently in state $j$ will produce $k$ offspring during the rest of its life in the present state and all other states it will visit thereafter. Then

$$p_j[l] = \left( \frac{b_j}{b_j + d_j - t_{jj}} \right)^l \left( \frac{d_j - t_{jj}}{b_j + d_j - t_{jj}} \right) \tag{2}$$

(i.e., the probability that there are $l$ birth-events followed by a single non-birth event which terminates the the stay in state $j$ either by a death event or a transition to another state), and

$$q_j[k] = p_j[k] \frac{d_j}{d_j - t_{jj}} + \sum_{l=0}^{k} p_j[l] \sum_{i \neq j} \left( q_i[k-l] \frac{t_{ij}}{d_j - t_{jj}} \right) \tag{3}$$

(i.e., the probability of producing $k$ offspring in state $j$ followed by a death event plus the probability of producing $l$ offspring in state $j$ and $k - l$ offspring during the rest of the individual's life after a transition to another state).

Let $f_j(z)$ and $g_j(z)$ denote the probability generating functions of the distributions $\{p_j[l]\}_{l \geq 0}$ and $\{q_j[k]\}_{k \geq 0}$. Then

$$f_j(z) = \frac{d_j - t_{jj}}{(1-z)b_j + d_j - t_{jj}} \tag{4}$$

and after some pretty straightforward calculations, also involving equation (4),

$$g_j(z)\big((1-z)b_j + d_j\big) = d_j + \sum_{\forall i} g_i(z) t_{ij}. \tag{5}$$

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Differentiation of equation (5) gives

$$R_j d_j - b_j = \sum_{\forall i} R_i t_{ij} \tag{6}$$

where we used that $g_j(1) = 1$ and $g_j'(1) = \mathcal{E}_j\{k\} = R_j$, which is the reproduction ratio of state $j$. Note that in particular $R_0$ is the well-known *basic reproduction ratio*. Define

$$\begin{aligned} \mathbf{R} &:= \begin{pmatrix} R_0 & \dots & R_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}, \quad \mathbf{B} := \begin{pmatrix} b_0 & \dots & b_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \\ \mathbf{D} &:= \begin{pmatrix} d_0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & d_n \end{pmatrix}, \quad \mathbf{T} := \begin{pmatrix} t_{00} & \dots & t_{0n} \\ \vdots & & \vdots \\ t_{n0} & \dots & t_{nn} \end{pmatrix} \end{aligned} \tag{7}$$

Since there is only one birth state, $\mathbf{R}$ is equal to the so-called next generation matrix. Equation (6) can be written in matrix notation as

$$\mathbf{R}(\mathbf{D} - \mathbf{T}) = \mathbf{B} \tag{8}$$

or equivalently

$$\mathbf{R} = \mathbf{B}(\mathbf{D} - \mathbf{T})^{-1} \tag{9}$$

which is possible because $\mathbf{D} - \mathbf{T}$ is strictly diagonally dominant and thus can be inverted.

Next, let $z_j$ denote the probability of the eventual extinction of the branching process starting in state $j$. Then, substitution of $z = z_0$ in equation (5) gives

$$z_j\big((1 - z_0)b_j + d_j\big) = d_j + \sum_{\forall i} z_i t_{ij} \tag{10}$$

where we used that $g_j(z_0) = z_j$ for all $j$. Let $\pi_j = 1 - z_j$ denote the probability of invasion starting from state $j$, then from equation (10) and equation (1) we get that

$$\pi_j(\pi_0 b_j + d_j) = \pi_0 b_j + \sum_{\forall i} \pi_i t_{ij}. \tag{11}$$

Define

$$\mathbf{\Pi} := \begin{pmatrix} \pi_0 & \dots & \pi_n \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix} \tag{12}$$

then equation (11) can be rewritten as

$$\mathbf{\Pi}(\pi_0 \mathbf{B} + \mathbf{D} - \mathbf{T}) = \pi_0 \mathbf{B}. \tag{13}$$

Right-multiplication with $(\mathbf{D}-\mathbf{T})^{-1}$, using equation (9), subsequently gives

$$\mathbf{\Pi}(\pi_0\mathbf{R}+\mathbf{I}) = \pi_0\mathbf{R} \tag{14}$$

or equivalently,

$$\mathbf{\Pi} = \pi_0\mathbf{R}(\pi_0\mathbf{R}+\mathbf{I})^{-1} \tag{15}$$

where $\mathbf{I}$ is the identity matrix. We can do this because $\pi_0\mathbf{R}+\mathbf{I}$ is the product of two non-singular matrices, namely $\pi_0\mathbf{B}+\mathbf{D}-\mathbf{T}$, which is strictly diagonally dominant, and $(\mathbf{D}-\mathbf{T})^{-1}$. Hence $\pi_0\mathbf{R}+\mathbf{I}$ is non-singular itself and can be inverted. Formal expansion of the right hand side of equation (15) gives

$$\mathbf{\Pi} = \pi_0\mathbf{R}\sum_{i=0}^{\infty}(-1)^i\pi_0^i\mathbf{R}^i \tag{16}$$

which converges whenever all eigenvalues of $\pi_0\mathbf{R}$ lie inside the unit circle in the complex plane, i.e., whenever $\pi_0 R_0 < 1$. Writing out equation (18) for the upper leftmost element (i.e., the only element that matters, really), we get

$$\pi_0 = \frac{\pi_0 R_0}{1+\pi_0 R_0} \tag{17}$$

i.e., $\pi_0 = 0$ or

$$\pi_0 = \frac{R_0 - 1}{R_0} \tag{18}$$

whenever the the latter is positive, i.e., whenever $R_0 > 1$. If $R_0 \leq 0$, then $\pi_0 = 0$ is the only solution. Thus, in conclusion, we have shown that

$$\pi_0 = \begin{cases} 0 & \text{if} \quad R_0 \leq 0 \\ \frac{R_0-1}{R_0} & \text{if} \quad R_0 > 1. \end{cases} \tag{19}$$

I would like to emphasize that this result is exact.