Prob. of extinction

$u(t,x) =$ prob. of being extinct at time $t \geq 0$ if pop. size at time zero is $x \geq 0$.

Satisfies Kolmogorov backward eqn.:

$$
① \begin{cases} \partial_t u = \mu(x) \partial_x u + \frac{\varepsilon}{2} \sigma^2(x) \partial_x^2 u \\ u(t,0) = 1 \\ u(t,\infty) = 0 \quad \text{(or some other appropriate bnd. cond.)} \end{cases}
$$

$\lim_{t \to \infty} u(t,x) =$ prob. of eventual extinction.

$$
② \quad 0 = \mu(x) u'(x) + \frac{\varepsilon}{2} \sigma^2(x) u''(x).
$$

Solution:

$$
③ \begin{cases} u(x) = 1 - \dfrac{\int_0^x G(\xi) d\xi}{\int_0^\infty G(\xi) d\xi} \leq 1 \\ G(x) := \exp\left\{ - \int_0^x \dfrac{\mu(\xi)}{\frac{\varepsilon}{2} \sigma^2(\xi)} d\xi \right\} > 0. \end{cases}
$$

$\boxed{\text{Corollary}}$:

(4) $u(x) < 1 \iff \int_0^\infty G(\xi)\,d\xi < \infty$.

$\boxed{\text{Example}}$ (logistic growth)

$b(x) = \beta x\,, \quad d(x) = \delta x^2$

$\mu(x) = b(x) - d(x)\,, \quad \sigma^2(x) = b(x) + d(x)$

$\Rightarrow \int_0^\infty G(x)\,dx = \infty \Rightarrow$ sure extinction.

$\boxed{\text{Example}}$ (lin. model)

$\mu(x) = (\beta - \delta)\,x\,, \quad \sigma^2(x) = (\beta + \delta)\,x$

$$\Rightarrow G(x) = \exp\left\{-\frac{2}{\varepsilon}\,\frac{\beta-\delta}{\beta+\delta}\cdot x\right\}$$

$$\Rightarrow \int_0^x G(\xi)\,d\xi = 1 - \frac{\varepsilon}{2}\,\frac{\beta+\delta}{\beta-\delta}\,e^{-\frac{2}{\varepsilon}\frac{\beta-\delta}{\beta+\delta}\cdot x}$$

$\Longrightarrow$ (next page)

$\boxed{\beta < \delta} \Longrightarrow \int_0^\infty g(\xi)\, d\xi = \infty$

$\Rightarrow \quad u(x) = 1$ (sure extinction)

$\boxed{\beta > \delta} \Rightarrow \int_0^\infty g(\xi)\, d\xi = 1 < \infty$

$$\Rightarrow u(x) = 1 - \int_0^x g(\xi)\, d\xi$$

$$= \frac{\varepsilon}{2} \frac{\beta+\delta}{\beta-\delta} e^{-\frac{2}{\varepsilon}\frac{\beta-\delta}{\beta+\delta}x} < 1$$

C.f. with ext. prob. in finite pop. lin. birth-death process:

Prob. ext. $(N) = \left(\frac{\delta}{\beta}\right)^N \quad \forall N \geqslant 0$.