PDE II
Demo 4

1. Prove that $W^{1,p}(\mathbb{R}^n) = W_0^{1,p}(\mathbb{R}^n)$.

2. Prove that if $u \in W^{1,p}(\Omega)$ and $\eta \in C_0^\infty(\Omega)$, then $u\eta \in W_0^{1,p}(\Omega)$.

3. Show by example that if $u \in L^1(\Omega)$ and there is $C > 0$ such that

$$||\Delta^h u||_{L^1(\Omega')} \leq C$$

for all $0 < |h| < \frac{1}{2}\text{dist}(\Omega', \partial\Omega)$, it does not necessarily follow that $u \in W^{1,1}(\Omega')$.

4. Suppose that $f_i, f \in L^2(\Omega), i = 1, 2, ...$ Prove that

$$f_i \to f \text{ strongly in } L^2(\Omega)$$

if and only if

$$f_i \rightharpoonup f \text{ weakly in } L^2(\Omega) \text{ and } \lim_{i\to\infty} ||f_i||_{L^2(\Omega)} = ||f||_{L^2(\Omega)}.$$

5. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Suppose that $f \in L^p(\Omega)$ for all $1 < p < \infty$ and

$$\lim_{p\to\infty} ||f||_{L^p(\Omega)} = M < \infty.$$

Show that $f \in L^\infty(\Omega)$ and $||f||_{L^\infty(\Omega)} = M$.