PDE II
Demo 3

1. Assume $0 < \beta < \gamma \leq 1$. Prove the interpolation inequality

$$||u||_{C^{0,\gamma}(\Omega)} \leq C||u||_{C^{0,\beta}(\Omega)}^{\frac{1-\gamma}{1-\beta}} ||u||_{C^{0,1}(\Omega)}^{\frac{\gamma-\beta}{1-\beta}}.$$

2. Prove that if $u \in W^{1,p}((0,1))$ for some $1 \leq p < \infty$, then u is equal *a.e.* to an absolutely continuous function (that is, for any $\varepsilon > 0$, there is $\delta > 0$ such that for any sequence of pairwise disjoint sub-intervals $[a_i, b_i] \subset (0,1)$ satisfying

$$\sum_i |b_i - a_i| < \delta,$$

we have

$$\sum_i |u(b_i) - u(a_i)| < \varepsilon.)$$

and u' (which exists *a.e.*) belongs to $L^p((0,1))$.

3. Suppose that Ω is a domain in $\mathbb{R}^n$ and $u \in W^{1,p}(\Omega)$ satisfies

$$Du = 0 \quad a.e. \text{ in } \Omega.$$

Prove that u is a constant *a.e.* in Ω.

4. Assume that $F : \mathbb{R} \to \mathbb{R}$ is C^1, with F' bounded. Suppose that $\Omega \subset \mathbb{R}^n$ is open, bounded, and $u \in W^{1,p}(\Omega)$ for some $1 < p < \infty$. Let $v = F(u)$. Show that

$$v_{x_i} = F'(u)u_{x_i} \quad \text{ and } F(u) \in W^{1,p}(\Omega).$$

5.

(i) Prove that if $u \in W^{1,p}(\Omega)$, then $|u| \in W^{1,p}(\Omega)$.

(ii) Prove that if $u \in W^{1,p}(\Omega)$, then $u^+ = \max(u, 0), u^- \in W^{1,p}(\Omega)$ and

$$Du^+ = \begin{cases} Du & a.e. \text{ on } \{u > 0\}; \\ 0 & a.e. \text{ on } \{u < 0\}. \end{cases}$$

(iii) Prove

$$Du = 0 \quad a.e. \text{ on } \{u = 0\}.$$