Department of Mathematics and Statistics
Measure and Integral
Exercise 7
Extra exercises, 2017

You may gain extra credit points by returning your solutions (in written) either to the lecturer or your instructor by Friday, March 10, 4 pm.

1. Find the limit
$$\lim_{k\to\infty}\int_0^\infty \frac{dx}{\sqrt{x+e^{k(x-1)}}}.$$

2. Let $(f_j)_{j=1}^\infty$ be a sequence of measurable functions $f_j\colon \mathbb{R}^n\to\mathbb{R}$ such that the sum $\sum_{j=1}^\infty |f_j|$ is integrable. Prove that
$$\lim_{j\to\infty}\int_{\mathbb{R}^n} f_j = 0.$$

3. Let $f\colon \mathbb{R}\to\mathbb{R}$ be integrable. Prove that the function $g\colon \mathbb{R}\to\mathbb{R}$,
$$g(x)=\int_{-\infty}^\infty f(t)\sin(xt)\,dt,\ \ x\in\mathbb{R},$$
is continuous.

4. Find the limit
$$\lim_{n\to\infty}\int_{-n}^n e^{-nx^2}\,dx.$$

5. Find the limit
$$\lim_{k\to\infty}\int_0^\infty \frac{\sin(x^k)}{x^{k-1}}\,dx.$$

6. Let $f\colon [0,1]\to\mathbb{R}$ be measurable. Prove that the functions
$$g_n(x)=\frac{\cos f(x)}{1+n\big(f(x)\big)^2}$$
are integrable over the interval $[0,1]$ and that the limit
$$a=\lim_{n\to\infty}\int_0^1 g_n(x)\,dx$$
exists. Find all the possible values of a, when f runs through all measurable functions $f\colon [0,1]\to\mathbb{R}$.