Department of Mathematics and Statistics
Measure and Integral
Exercise 4
15-17.2.2017

1. If $(A_n)_{n=1}^\infty$ is a sequence of subsets of X, we denote

$$\liminf_{n\to\infty} A_n = \bigcup_{n=1}^\infty \left(\bigcap_{k=n}^\infty A_k \right) \quad \text{and} \quad \limsup_{n\to\infty} A_n = \bigcap_{n=1}^\infty \left(\bigcup_{k=n}^\infty A_k \right).$$

[$\liminf$ = *limes inferior*, $\limsup$ = *limes superior*]

Let (X, Γ, μ) be a measure space and $A_j \in \Gamma$ for all $j \in \mathbb{N}$.

(a) Prove that

$$\liminf_{j\to\infty} A_j \in \Gamma, \quad \limsup_{j\to\infty} A_j \in \Gamma$$

and

$$\mu(\liminf_{j\to\infty} A_j) \leq \lim_{k\to\infty} \left(\inf_{j\geq k} \mu(A_j) \right).$$

(b) Prove that

$$\mu(\limsup_{j\to\infty} A_j) \geq \lim_{k\to\infty} \left(\sup_{j\geq k} \mu(A_j) \right),$$

if $\mu\left(\bigcup_{j=1}^\infty A_j\right) < \infty$.

(c) Prove the so-called Borel-Cantelli Lemma: If $\sum_{j=1}^\infty \mu(A_j) < \infty$, then $\mu(\limsup_{j\to\infty} A_j) = 0$, that is, almost all points belong to at most finitely many A_j's.

2. Suppose that $A_1 \subset A_2 \subset A_3 \subset \cdots \subset \mathbb{R}^n$. Prove that

$$m^*\left(\bigcup_{i=1}^\infty A_i \right) = \lim_{i\to\infty} m^*(A_i).$$

3. Prove that the set $A = \{(x,y) \in \mathbb{R}^2 \colon x > 1 \text{ ja } 0 \leq yx^2 < 1\}$ is measurable.

4. Let $A \subset \mathbb{R}^n$ and $f \colon A \to \mathbb{R}$. Let

$$A_r = \{x \in A \colon f(x) > r\},$$

when $r \in \mathbb{R}$. Prove: if $m_n^*(A_0) > 0$, there exists $r > 0$ such that $m_n^*(A_r) > 0$.

5. Let $A \subset \mathbb{R}^n$ and $f \colon A \to \mathbb{R}$. Suppose that there exist sets $B_1, B_2, \ldots$ such that $A = \cup_{i=1}^\infty B_i$ and the restriction $f|B_i$ is measurable for every i. Prove that f is measurable.

6. Let $A_k \subset [0,1]$, $k = 1, 2, \ldots$, be measurable. Suppose that

$$m(A_k) > \frac{2^k - 1}{2^k}.$$

for all $k \in \mathbb{N}$. Prove that the intersection $\bigcap_{k=1}^{\infty} A_k$ is non-empty.