Department of Mathematics and Statistics
Measure and Integral
Exercise 3
8-10.2.2014

1. True or false (and why)?
   (a) If $A \subset \mathbb{R}^n$ is open, then its boundary ∂A is measurable.
   (b) If $B \subset \mathbb{R}^n$ is an arbitrary set, then ∂B is measurable.
   (c) If $A \subset \mathbb{R}$ is open and bounded, then $m_1(\partial A) = 0$.

2. Let $E \subset \mathbb{R}^n$ be measurable and $A \subset \mathbb{R}^n$. Prove that
$$m^*(A \cup E) + m^*(A \cap E) = m^*(A) + m^*(E).$$

3. Let $A = \{(x, y) \in \mathbb{R}^2 \colon x \in [0, 1], y \in [0, 1] \setminus \mathbb{Q}\}$. Prove that A is measurable and compute $m_2(A)$.

4. Prove that $\{(x, y) \in \mathbb{R}^2 \colon x \in \mathbb{Q} \text{ or } y \notin \mathbb{Q}\}$ is measurable.

5. Let
$$\mathcal{F}_\sigma = \{\bigcup_{i=1}^{\infty} F_i \colon F_i \subset \mathbb{R}^n \text{ is closed } \forall i\}$$
and
$$\mathcal{G}_\delta = \{\bigcap_{i=1}^{\infty} G_i \colon G_i \subset \mathbb{R}^n \text{ is open } \forall i\}.$$
Prove that $\mathcal{F}_\sigma \subset \operatorname{Bor} \mathbb{R}^n$ and $\mathcal{G}_\delta \subset \operatorname{Bor} \mathbb{R}^n$, where $\operatorname{Bor} \mathbb{R}^n$ is the family of all Borel subsets of $\mathbb{R}^n$.

6. Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ be continuous and $A \subset \mathbb{R}^n$ is closed. Prove that $fA \in \mathcal{F}_\sigma$.