Department of Mathematics and Statistics
Measure and integral
Exercises 2
1-3.2.2017

1. True or false (and why)?
   (a) If $m^*(A) > 0$, then A contains a non-empty open set.
   (b) If $m^*(A) < \infty$, then A is bounded.

2. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, $f(x_1, x_2) = (t_1x_1, t_2x_2)$, where $t_1, t_2 \in \mathbb{R}$. Prove that
$$m_2^*(fA) = |t_1t_2|m_2^*(A)$$
for all $A \subset \mathbb{R}^2$.

3. We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *Lipschitz* if there exists a constant $L > 0$ such that $|f(x) - f(y)| \le L|x - y|$ for all $x, y \in \mathbb{R}$. Prove: If $A \subset \mathbb{R}$ is of zero measure and $f\colon \mathbb{R} \to \mathbb{R}$ is Lipschitz, then also the image fA is measurable and $m(fA) = 0$.

4. Prove that a set $E \subset \mathbb{R}^n$ is measurable if and only if
$$m^*(S \cup U) = m^*(S) + m^*(U)$$
for all $S \subset E$ and $U \subset \mathbb{R}^n \setminus E$.

5. Prove that a set $E \subset \mathbb{R}^n$ is measurable if and only if
$$m^*(I) = m^*(I \cap E) + m^*(I \setminus E)$$
for every open n-interval I. [You can use the fact that $m^*(I) = \ell(I)$ for an n-interval I.]

6. Let $A \subset \mathbb{R}^n$. (a) Prove that for all $\varepsilon > 0$ there exists an open set $B \subset \mathbb{R}^n$ such that $A \subset B$ and
$$m_n^*(B) \le m_n^*(A) + \varepsilon.$$
(b) Prove that there exist open sets $B_k \subset \mathbb{R}^n$ such that
$$A \subset \bigcap_{k=1}^{\infty} B_k \quad \text{and} \quad m_n^*\big(\cap_{k=1}^{\infty} B_k\big) = m_n^*(A).$$