Department of Mathematics and Statistics
Measure and integral
Exercises 1
25-27.1.2017

1. (a) Find $\inf E$ and $\sup E$ if $E = \{\frac{1}{\log x} \in \mathbb{R} \colon x > 1\}$.
   (b) Suppose that $\emptyset \neq B \subset A \subset \mathbb{R}$. Prove that
   $$\inf A \leq \inf B \leq \sup B \leq \sup A.$$
   (c) Let $\emptyset \neq A \subset \mathbb{R}$ and $-2A = \{-2x \colon x \in A\}$. Prove that
   $$\inf(-2A) = -2 \sup A.$$

2. Let $V_1, \dots, V_k \subset \mathbb{R}^n$ be open and let $F_1, \dots, F_k \subset \mathbb{R}^n$ be closed subsets of $\mathbb{R}^n$. Prove that $\cap_{j=1}^k V_j$ is open and $\cup_{j=1}^k F_j$ is closed. Also find examples of the following phenomena: (a) $V_j \subset \mathbb{R}$ is open for every $j \in \mathbb{N}$, but the intersection $\cap_{j=1}^\infty V_j$ is not an open set; (b) $F_j \subset \mathbb{R}$ is closed for every $j \in \mathbb{N}$, but the union $\cup_{j=1}^\infty F_j$ is not a closed set.

3. Let I be an uncountable set and $a_i > 0$ for every $i \in I$. Prove that
   $$\sum_{i \in I} a_i := \sup_{J \subset I \text{ finite}} \sum_{j \in J} a_j = +\infty.$$

4. Let
   $$\mathcal{B} = \{B^n(x, r) \subset \mathbb{R}^n \colon x = (x_1, \dots, x_n) \in \mathbb{Q}^n, r \in \mathbb{Q}, r > 0\}.$$
   Prove that $\mathcal{B}$ is countable. [In other words, $\mathcal{B}$ is a collection of open balls $B^n(x, r)$ in $\mathbb{R}^n$ such that the coordinates of the centers x are rational numbers and the radii r are positive rational numbers.]

5. Let A be the set of all rational numbers on the interval $[0, 1]$, that is $A = [0, 1] \cap \mathbb{Q}$.
   (a) Let $I_i =]a_i, b_i[$, $i = 1, \dots, k$, be open intervals such that
   $$A \subset \bigcup_{i=1}^k I_i.$$
   Prove that
   $$\sum_{i=1}^k (b_i - a_i) \geq 1$$
   for every (finite) $k \in \mathbb{N}$.

(b) Prove that, for every $\varepsilon > 0$, there exist open intervals $I_i =]a_i, b_i[$, $i \in \mathbb{N}$, such that

$$A \subset \bigcup_{i=1}^{\infty} I_i \quad \text{and} \quad \sum_{i=1}^{\infty} (b_i - a_i) < \varepsilon.$$

6. Let $A \subset \mathbb{R}^n$, $y \in \mathbb{R}^n$, and $k > 0$. Define

$$A + y = \{x + y \colon x \in A\} \quad \text{and} \quad kA = \{kx \colon x \in A\}.$$

Prove:

$$m_n^*(A + y) = m_n^*(A) \quad \text{and} \quad m_n^*(kA) = k^n m_n^*(A).$$