TRAVELING SALESMAN THEOREMS AND THE CAUCHY TRANSFORM

TUOMAS ORPONEN

ABSTRACT. These are the lecture notes for the course *Geometric measure theory and singular integrals*, given in Spring 2017 at the University of Helsinki. They contain the constructive part of P. Jones' L^∞ traveling salesman theorem in the plane, following the book of Bishop and Peres, and two proofs of an L^1 traveling salesman theorem for doubling measures, due to Badger and Schul (the first proof assumes a quantitative form of non-atomicity from the measure, and follows an argument of Tolsa).

As an application of the traveling salesman theorems, the notes contain a proof of the Mattila-Melnikov-Verdera theorem from 1996, on the Cauchy transform and uniform rectifiability. Several additional topics are also discussed:

- David's theorem, stating that non-atomic measures with bounded Cauchy transform have linear growth,
- the Denjoy conjecture (aka Calderón's theorem), stating that positive-length subsets of rectifiable curves are non-removable for bounded analytic functions,
- and finally the fact that sufficiently irregular sets, including the four-corners Cantor set, are removable for bounded analytic functions.

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1. Introduction

Remark 1.1. The material for the lecture notes has been gathered and combined from various sources, see the list of references. The main sources are the books of Bishop-Peres [2], Falconer [9], Mattila [12], and Tolsa [17], and the article of Badger-Schul [1].

These lecture notes have two main goals. First, to describe and prove various "traveling salesman theorems", see Section 2.2 for an overview. Second, to explore the connections between rectifiability, the Cauchy transform, and removability. In particular, we prove the following theorem of P. Mattila, M. Melnikov and J. Verdera [13] from 1996:

Theorem 1.2. *Let $E \subset \mathbb{C}$ be a 1-AD regular set such that the Cauchy transform associated to $\mathcal{H}^1|_E$ is bounded on $L^2(\mathcal{H}^1|_E)$. Then, the set E is uniformly 1-rectifiable.*

The Cauchy integral operator associated to a Radon measure μ is, formally speaking, the object

$$\mathcal{C}_\mu f(z) = \int \frac{f(w)}{z-w}\, d\mu w.$$

For more details and results, see Section 6. A "1-AD-regular set" is short for a *1-Ahlfors-David regular set*, defined below:

Definition 1.3 (AD regularity)**.** Let $0 \leq s \leq n$. A Borel set $E \subset \mathbb{R}^n$ is called *s-Ahlfors-David regular*, or *s-AD regular* in short, if there is a constant $M \geq 1$ such that

$$\frac{r^s}{M} \leq \mathcal{H}^s(E \cap B(x,r)) \leq Mr^s, \qquad x \in E,\ 0 < r \leq \operatorname{diam}(E).$$

More generally, a Borel measure μ is called s-AD regular, if $r^s/M \leq \mu(B(x,r)) \leq Mr^s$ for $x \in \operatorname{spt}\mu$ and $0 < r \leq \operatorname{diam}(\operatorname{spt}\mu)$; thus, a Borel set E is s-AD regular, if and only if $\mathcal{H}^s|_E$ is s-AD regular.

The distinction between AD regular sets and measures is mostly semantic: if μ is s-AD regular with $0 \leq s \leq n$, then $\mu = \mathcal{H}^s|_E$ for an s-AD regular set E. Since only 1-AD regular sets and measures will be considered in these lecture notes, I will abbreviate

$$\text{AD-regular } = \text{1-AD-regular.}$$

Here is one possible definition of uniform 1-rectifiability:

Definition 1.4 (Uniform rectifiability)**.** Let $n \geq 2$. A set $E \subset \mathbb{R}^n$ is called *uniformly 1-rectifiable*, if there is a constant $C \geq 1$ with the following property: for every ball $B \subset \mathbb{R}^n$, the intersection $E \cap B$ can be covered by a continuum Γ_B satisfying $\mathcal{H}^1(\Gamma_B) \leq C \operatorname{diam}(B)$. A Radon measure μ is called uniformly 1-rectifiable, if $\operatorname{spt}\,\mu$ is uniformly 1-rectifiable.

Remark 1.5. The basic example of a uniformly 1-rectifiable set is an AD regular continuum. In fact, if E is 1-AD regular to begin with (as in Theorem 1.2), then it is known (see [7], the discussion at the end of p. 14) that E is uniformly 1-rectifiable, if and only if E is contained in an AD regular continuum. In the plane, and for compact sets, this equivalence is quite easy to prove, even without the *a priori* AD regularity assumption.

Exercise 1.6. Let $E \subset \mathbb{R}^2$ a uniformly 1-rectifiable compact set. Prove that there exists an AD regular continuum $\Gamma \supset E$ with $\mathrm{diam}(\Gamma) \sim \mathrm{diam}(E)$, where all the implicit constants only depend on C. *Hint*: read Section 3 first.

Remark 1.7. At least two essentially different proofs of Theorem 1.2 are now available: the original from [13], based on *curvature*, and then a more recent one based on the notion of *reflectionless measures*, due to B. Jaye and F. Nazarov [10]. In these lecture notes, I take the original route; that said, many of the details are gathered and pieced together from resources more recent than [13].

Remark 1.8. Theorem 1.2 says that if μ is *a priori* AD regular, then the L^2-boundedness of $\mathcal{C}_\mu$ on $L^2(\mu)$ implies that μ is uniformly 1-rectifiable. It is fair to ask, whether the *a priori* regularity assumption is sensible. G. David [5] has shown that if μ is *non-atomic* to begin with, and $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$,[1] then

$$\mu(B(x,r)) \leq Cr, \qquad x \in \mathbb{R}^2,\ r > 0,$$

where $C \geq 1$ depends on the constants in the L^2-boundedness; on the other hand, $\mathcal{C}_{\delta_0}$ is nearly trivially bounded on $L^2(\delta_0)$. I postpone the proof of David's result to Section 6, see Proposition 6.9.

The inequality $\mu(B(x,r)) \leq Cr$ from David's result is, of course, the "upper" inequality required for AD regularity. The lower regularity is definitely **not** necessary for the $L^2(\mu)$-boundedness of $\mathcal{C}_\mu$. For instance, fix any s-AD regular set $E \subset \mathbb{R}^2$ with $s > 1$ and $0 < \mathcal{H}^s(E) < \infty$, and let $\mu = \mathcal{H}^s|_E$. A simple computation shows that

$$\int \frac{d\mu w}{|z-w|^q} \leq C, \qquad z \in \mathbb{R}^2,$$

for any $0 \leq q < s$. Now, fix some such $q < s$, and let $p < \infty$ be the dual exponent. Then, cheating a little bit (to be precise, you should do the following for the ϵ-truncations)

$$\begin{aligned}\|\mathcal{C}_\mu(f)\|_{L^p(\mu)}^p = \int |\mathcal{C}_\mu(f)|^p \, d\mu &\leq \int \left(\int \frac{|f(w)|}{|z-w|} \, d\mu w \right)^p \, d\mu z \\ &\leq \int \int |f(w)|^p \, d\mu w \left(\int \frac{d\mu w}{|z-w|^q} \right)^{p/q} \, d\mu z \lesssim_{E,s} \int |f(w)|^p \, d\mu w.\end{aligned}$$

So, $\mathcal{C}_\mu$ is bounded on $L^p(\mu)$. It is also easy to see that $\mathcal{C}_\mu$ is bounded on $L^q(\mu)$: if $f \in L^q(\mu)$ and $g \in L^p(\mu)$, then

$$\left| \int \mathcal{C}_\mu(f) \cdot g \, d\mu \right| = \left| \int f \cdot \mathcal{C}_\mu(g) \, d\mu \right| \leq \|f\|_{L^q(\mu)} \|\mathcal{C}_\mu(g)\|_{L^p(\mu)} \lesssim \|f\|_{L^q(\mu)} \|g\|_{L^p(\mu)}$$

by the previous computation, and now $\|\mathcal{C}_\mu\|_{L^q(\mu) \to L^q(\mu)} < \infty$ by duality. Finally, standard Marcinkiewicz interpolation gives $\|\mathcal{C}_\mu\|_{L^2(\mu) \to L^2(\mu)} < \infty$.

So, in conclusion, the upper AD regularity condition $\mu(B(x,r)) \lesssim r$ is **necessary** in Theorem 1.2, for non-atomic measures, whereas the lower inequality $\mu(B(x,r)) \gtrsim r$ is there **to make life interesting**. For measures decaying much more rapidly than $O(r)$, the problem is too easy.

[1] In the the usual sense that all ϵ-truncations are uniformly bounded on $L^2(\mu)$, see Section 6.

2. Uniform rectifiability and β-numbers

To prove Theorem 1.2, one needs to develop the theory of uniformly 1-rectifiable sets. One of the seminal results in this theory was the *Analyst's traveling salesman theorem* of P. Jones from 1990 [11], which provided a useful "multi-scale" characterisation of uniform 1-rectifiability (the terminology "uniformly rectifiable" was coined shortly afterwards by G. David and S. Semmes in another seminal paper [7]). Jones' characterisation is formulated in terms of β-numbers, which I will now discuss.

2.1. **Various β-numbers.** Let μ be a Radon measure on $\mathbb{R}^n$. In the typical application, $\mu = \mathcal{H}^1|_E$ for a set E with positive and σ-finite 1-dimensional measure. For $p \in [1, \infty)$, a compact set B (which will always be a ball or a cube) and a straight line $\ell \subset \mathbb{R}^n$, write

$$\beta_{\mu,p}(B, \ell) := \left[\int_B \left(\frac{\operatorname{dist}(x, \ell)}{\operatorname{diam}(B)} \right)^p \frac{d\mu x}{\mu(B)} \right]^{1/p},$$

where we agree that $\beta_{\mu,p}(B, \ell) = 0$, if $\mu(B) = 0$. Then, define

$$\beta_{\mu,p}(B) := \inf_{\text{lines } \ell} \beta_{\mu,p}(B, \ell).$$

It is clear from Hölder's inequality that

$$\beta_{\mu,p_1}(B) \leq \beta_{\mu,p_2}(B), \qquad 1 \leq p_1 \leq p_2 < \infty. \tag{2.1}$$

How about $p = \infty$? The definition practically writes itself: again, for a line $\ell \subset \mathbb{R}^n$, define the auxiliary number

$$\tilde{\beta}_{\mu,\infty}(B, \ell) := \mu - \operatorname*{ess\,sup}_{x \in B} \frac{\operatorname{dist}(x, \ell)}{\operatorname{diam}(B)},$$

and then set

$$\tilde{\beta}_{\mu,\infty}(B) := \inf_{\text{lines } \ell} \tilde{\beta}_{\mu,\infty}(B, \ell).$$

It is clear from (2.1) that the numbers $\beta_{p,\mu}(B)$ tend to a limit as $p \to \infty$, and the limit is bounded by $\tilde{\beta}_{\mu,\infty}(B)$.

Exercise 2.2. Prove or disprove:

$$\tilde{\beta}_{\mu,\infty}(B) = \lim_{p \to \infty} \beta_{\mu,p}(B).$$

There is another fairly natural definition for $\tilde{\beta}_{\mu,\infty}$, which will be used more in in these lecture notes (both for historical reasons, and for convenience). For any set $E \subset \mathbb{R}^n$ (such as $E = \operatorname{spt} \mu$), write

$$\beta_{E,\infty}(B, \ell) := \sup_{x \in B \cap E} \frac{\operatorname{dist}(x, \ell)}{\operatorname{diam}(B)}.$$

The number $\beta_{E,\infty}(B)$ is then defined in the obvious way. Thanks to the following inequalities, it makes little practical difference, which convention for β_∞ is used:

$$\tilde{\beta}_{\mu,\infty}(B) \leq \beta_{\operatorname{spt} \mu,\infty}(B) \lesssim_\lambda \tilde{\beta}_{\mu,\infty}(\lambda B), \quad \lambda > 1, \tag{2.3}$$

where

$$\lambda B = \{x : \operatorname{dist}_\infty(x, B) \leq (\lambda - 1) \operatorname{diam}_\infty(B)\}, \quad \lambda \geq 1. \tag{2.4}$$

Here $\operatorname{dist}_\infty$ and $\operatorname{diam}_\infty$ refer to distance and diameter in the L^∞-distance $\|x - y\|_\infty = \max\{|x_i - y_i| : 1 \leq i \leq n\}$. This detail will be convenient in the sequel, where the notation is mostly applied to cubes: with the current definition, λQ remains a cube for all $\lambda \geq 1$, see Figure 1.

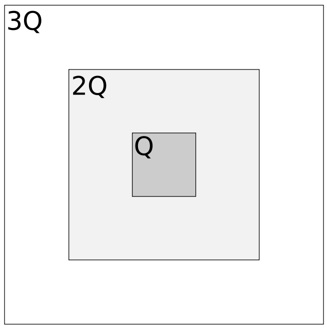

FIGURE 1. The cubes $Q, 2Q$ and $3Q$ with our convention of "λE".

2.2. An overview of traveling salesman theorems.

2.2.1. *Jones' traveling salesman theorem for β_∞-numbers.* As mentioned above, this is where it all started in 1990:

Theorem 2.5 (Jones). *Let $\mathcal{D}$ be the family of* ***closed***[2] *dyadic cubes in $\mathbb{R}^n$, and let $E \subset \mathbb{R}^n$ be a compact set satisfying*

$$\beta_\infty^2(E) := \sum_{Q \in \mathcal{D}} \beta_{E,\infty}^2(2Q)\ell(Q) < \infty. \tag{2.6}$$

Then, for any $\delta > 0$, there exists a compact connected set $\Gamma \subset \mathbb{R}^n$ such that $E \subset \Gamma$, and

$$\mathcal{H}^1(\Gamma) \leq (1 + \delta)\,\mathrm{diam}(E) + C_\delta \beta_\infty^2(E).$$

The proof of Jones' theorem in the plane is contained in Section 4. In fact, Jones also proved the converse for $n = 2$, using complex analysis. The result was generalised to higher dimensions, with a different, geometric proof, by K. Okikiolu [15] a bit later: for rectifiable curves Γ of finite length, the sum $\beta_\infty^2(\Gamma)$ is bounded by $\lesssim \mathcal{H}^1(\Gamma)$. Jones also observed in [11] that if, in place of (2.6), the β_∞-numbers satisfy the following *Carleson condition*,

$$\sum_{Q \subset R} \beta_{E,\infty}^2(2Q)\ell(Q) \lesssim \ell(R), \qquad R \in \mathcal{D}, \tag{2.7}$$

with implicit constants independent of R of course, then E can be covered by an AD regular continuum: in particular, E is uniformly 1-rectifiable. Note that the Carleson condition (2.7) can sometimes hold, even if the full sum in (2.6) diverges: this is, for instance, the case for unbounded AD regular curves (unless they happen to be lines, or otherwise sufficiently flat at infinity).

[2]This distinction makes no different in this theorem, but it will be useful later, and I want to keep the same notation everywhere.

2.2.2. *Traveling salesman theorems for β_p-numbers.* The L^p-versions of the β-numbers do not make sense for sets, unless there is some canonical measure supported on the set. For measures μ, however, it is reasonable to ask whether a condition for the $\beta_{\mu,p}$-numbers $p \in [1, \infty]$, analogous to either (2.6) or (2.7) gives some geometric information about the support of μ. The answer is positive, to a certain extent, and this was one of the main results in David and Semmes' paper [7], where the notion of uniform rectifiability was first introduced.

Theorem 2.8 (David-Semmes). *Assume that $E \subset \mathbb{R}^n$ is a 1-AD regular set, let $\mu := \mathcal{H}^1|_E$, and let $p \in [1, \infty]$. Assume that the $\beta_{\mu,p}$-numbers satisfy the Carleson condition*

$$\sum_{Q \subset R} \beta_{\mu,p}^2(2Q)\ell(Q) \lesssim \ell(R), \qquad R \in \mathcal{D}. \tag{2.9}$$

Then E is uniformly 1-rectifiable (in fact, $\operatorname{spt} \mu$ *can be covered by a 1-AD regular continuum).*

Using the inequalities (2.3), the case $p = \infty$ reduces easily to the previous result of Jones, but the cases $p \in [1, \infty)$ are *a priori* harder, because a "cube-wise" comparison "$\beta_{\mu,p}(Q) \sim \tilde{\beta}_{\mu,\infty}(Q)$" is not true for $p < \infty$.[3] However, for 1-AD regular measures (and more generally "smooth" measures, to be introduced in Section 5), these cases can also be reduced to the case $p = \infty$ via the following trick (Theorem 7.52 in Tolsa's book [17]):

Theorem 2.10 (Tolsa). *Let μ be a smooth Radon measure in $\mathbb{R}^n$. Then*

$$\sum_{Q \subset R} \beta_{\operatorname{spt} \mu,\infty}^2(2Q)\ell(Q) \lesssim \sum_{Q \subset 2R} \beta_{\mu,p}^2(3Q)\ell(Q)$$

for any cube $R \in \mathcal{D}$, and any $p \in [1, \infty)$.

Recall that a Radon measure μ on $\mathbb{R}^n$ is called *doubling*, if there exists a constant $D_\mu \geq 1$ such that

$$\mu(B(x, 2r)) \leq D_\mu \mu(B(x, r)), \qquad x \in \operatorname{spt} \mu, \ r > 0.$$

The constant D_μ is called the *doubling constant of* μ. It turns out that Theorem 2.9 holds for all doubling measures,[4] and the following results are some of the main topics of these lecture notes. They are due to M. Badger and R. Schul [1] from 2016.

Theorem 2.11 (Badger-Schul). *Let μ be a doubling measure on $\mathbb{R}^n$ with compact support $E :=$* $\operatorname{spt} \mu$*, let $p \in [1, \infty)$, and assume that the numbers $\beta_{\mu,p}$ satisfy*

$$\beta_p^2(\mu) := \sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset \lambda E}} \beta_{\mu,p}^2(2Q)\ell(Q) < \infty,$$

where $\lambda = \lambda_n \geq 1$ is a sufficiently large constant, and $\mathcal{D}_E = \{Q \in \mathcal{D} : Q \cap E \neq \emptyset\}$. Then E can be covered by a continuum $\Gamma \subset \mathbb{R}^n$ with

$$\mathcal{H}^1(\Gamma) \lesssim_{D_\mu, n} \operatorname{diam}(E) + \beta_p^2(\mu).$$

[3] One can check that if E is AD regular, and $\mu = \mathcal{H}^1|_E$, then $\beta_{E,\infty}(2Q) \lesssim \beta_{\mu,1}(5Q)^{1/2}$. This bound is sometimes quite useful, but not good enough for direct application to the traveling salesman problem.

[4] It is an open research topic, which are the "minimal" *a priori* assumptions on a measure, so that summability of the β-numbers implies rectifiability, as in Theorem 2.9. Without any *a priori* assumptions, the situation does not look very promising at the moment [14].

Corollary 2.12. *Assume that μ is a doubling measure on $\mathbb{R}^n$ and $p \in [1, \infty]$. If the numbers $\beta_{\mu,p}$ satisfy the Carleson condition*

$$\sum_{\substack{Q \in \mathcal{D}_{\operatorname{spt}\mu} \\ Q \subset R}} \beta_{\mu,p}^2(2Q)\ell(Q) \lesssim \ell(R), \qquad R \in \mathcal{D},$$

then μ is uniformly 1-rectifiable.

Proof. The case $p = \infty$ already follows from the work of Jones and does not require doubling from μ. The case $p \in [1, \infty)$ uses Theorem 2.11 as follows. Fix a ball $B \subset \mathbb{R}^n$. By Theorem 2.11, the intersection $B \cap (\operatorname{spt}\mu)$ can be covered by a continuum Γ_B of length

$$\mathcal{H}^1(\Gamma_B) \lesssim \operatorname{diam}(B \cap (\operatorname{spt}\mu)) + \sum_{\substack{Q \in \mathcal{D}_{\operatorname{spt}\mu} \\ Q \subset \lambda B}} \beta_{\mu,p}^2(2Q)\ell(Q) \lesssim \operatorname{diam}(B).$$

The last inequality follows from the Carleson condition, and the fact that λB can be covered by $\lesssim 1$ cubes $R \in \mathcal{D}$ with $\ell(R) \sim \operatorname{diam}(B)$. □

Section 5 contains two proofs for Theorem 2.11: The first one only works for "smooth" measures (which means "doubling + quantitatively non-atomic"), in which case the result can be reduced to Jones' L^∞ traveling salesman theorem via Tolsa's trick, Theorem 2.10. The second one is the original proof by Badger and Schul, and works for all doubling measures.

3. Preliminaries on compact and (mostly) connected sets

This section is not strictly necessary for the sequel, but it would be odd to read about traveling salesman theorems without knowing the material here. I take the following result for granted (see [8, Exercise 6.3.12]):

Theorem 3.1. *A connected set $\Gamma \subset \mathbb{R}^n$ with finite length is* arcwise connected*: for every pair $x, y \in \Gamma$ there exists an injective curve $\psi([0,1]) \subset \Gamma$ such that $\psi(0) = x$ and $\psi(1) = 1$.*

The rest of the material from this section is from Falconer's book [9]. Let's recall the Hausdorff metric on non-empty compact subsets of $\mathbb{R}^n$. For $K \subset \mathbb{R}^n$ and $\delta > 0$, write $K(\delta) := \{x \in R^n : \operatorname{dist}(x, K) < \delta\}$. Then, for non-empty compact sets $K_1, K_2 \subset \mathbb{R}^n$, let

$$d_H(K_1, K_2) = \inf\{\delta > 0 : K_1 \subset K_2(\delta) \text{ and } K_2 \subset K_1(\delta)\}.$$

Then (exercise, if this is news) d_H is a metric on non-empty compact subsets of $\mathbb{R}^n$. A very useful result is that "the set of compact sets is a compact space" (at least almost):

Theorem 3.2 (Blaschke selection theorem)**.** *Let $\mathcal{F}$ be an infinite family of non-empty compact sets in $\mathbb{R}^n$, all lying in some fixed closed ball $B \subset \mathbb{R}^n$. Then, there exists a sequence of* ***distinct*** *sets $\{K_j\}_{j \in \mathbb{N}} \subset \mathcal{F}$, and a non-empty compact set $K \subset B$ such that $K_j \to K$ in the Hausdorff metric.*

Remark 3.3. The conclusion would be pretty obvious without the requirement that that the sets K_j be distinct. Also, the family $\mathcal{F} = \{\{k\} \subset \mathbb{R} : k \in \mathbb{N}\}$ shows that the hypothesis involving B is necessary.

Proof. Induction: let $\{K_j^1\} \subset \mathcal{F}$ be **any** sequence of distinct sets, and assume that the sequence $\{K_j^m\}$ has already been defined for some $m \geq 1$. Define a subsequence $\{K_j^{m+1}\} \subset \{K_j^m\}$ as follows. Cover B by a finite number $\mathcal{B}^m$ of balls of diameter $1/m$. Then, every set K_j^m intersects every ball in some finite sub-collection $\mathcal{B}_j^m$; there are only finitely many different sub-collections, and infinitely many distinct sets, so there must be a **fixed** sub-collection $\tilde{\mathcal{B}}^m$ such that $\tilde{\mathcal{B}}^m = \mathcal{B}_j^m$ for infinitely many indices j. Define the subsequence $\{K_j^{m+1}\}$ by picking only those indices j. Then, it is clear that

$$d_H(K_i^{m+1}, K_j^{m+1}) \leq \frac{2}{m}, \quad i, j \in \mathbb{N},$$

because K_i^{m+1} and K_j^{m+1} both intersect all the balls in $\tilde{\mathcal{B}}^m$, and are covered by them.

Because $\{K_j^{m+p}\}$ is always a subsequence of $\{K_j^m\}$, it follows that

$$d_H(K_j^{m+p}, K_i^m) \leq \frac{2}{m} \tag{3.4}$$

for all i, j and $p \geq 0$. Now, let $K_m := K_m^m$. From (3.4), one sees that $d_H(K_m, K_{m+p}) \leq 2/m$, which implies that $\{K_m\}_{m \in \mathbb{N}}$ is a Cauchy sequence in d_H. So, it remains to show that d_H is a complete metric.

Define

$$K := \bigcap_{m \geq 0} \overline{\bigcup_{k \geq m} K_k} =: \bigcap_{m \geq 0} E_m.$$

This is clearly a non-empty compact set. Further, since $K_{m+p} \subset K_m(2/m)$ for all $p \geq 0$, it follows that $E_{m+p} \subset K_m(2/m)$ for $p \geq 0$, and consequently

$$K \subset K_m(2/m), \qquad m \geq 0.$$

If the converse inclusion were also true, then $d_H(K, K_m) \leq 3/m$, and the proof would be complete. So, it suffices to check that the converse inclusion is true.

Pick $x \in K_m$. Then $x \in K_{m+p}(2/m)$ for all $p \geq 0$, so also $x \in E_{m+p}(2/m)$. Now, it suffices to choose a sequence $\{y_{m+p}\}$ with $y_{m+p} \in E_{m+p}$ with $|x - y_{m+p}| \leq 2/m$. The sequence has a subsequence convergent to a point $y \in K$ satisfying $|x - y| \leq 2/m$. This proves that $K_m \subset K(2/m)$. □

A *tree* T is a continuum without loops: that is, for every pair $x, y \in T$, there is a unique path $\gamma([0,1]) \subset \Gamma$ with $\gamma(0) = x$ and $\gamma(1) = y$. For $x, y \in T$, let $d_\gamma(x, y)$ be the *path distance* between x, y: $d_\gamma(x, y) = \mathcal{H}^1(\gamma_{x,y})$, where $\gamma_{x,y}$ is the unique path joining x to y in T. This is a metric on T, and satisfies $d_\gamma(x, y) \geq |x - y|$. The diameter of a set $K \subset T$ in the d_γ metric is denoted by $\operatorname{diam}_\gamma(K)$.

For the purposes below, the key feature of trees is the following: they can be easily chopped into pieces of smaller diameter **preserving connectedness**. That would not be so easy for arbitrary continuums.

Lemma 3.5 (Tree-chopping lemma)**.** *Let $T \subset \mathbb{R}^n$ be a tree with finite length. Then, given $\delta > 0$, we can express T as the union of $\mathcal{H}^1$-essentially disjoint sub-trees $T_1, \ldots, T_m$ with the following properties:*

(a) $\operatorname{diam}(T_k) \leq \min\{\delta, \mathcal{H}^1(T_k)\}$ *for* $1 \leq k \leq m$,
(b) $m \lesssim \mathcal{H}^1(T)/\delta + 1$.

Remark 3.6. The inequality $\operatorname{diam}(T_k) \leq \mathcal{H}^1(T_k)$ follows simply from the fact that T_k is connected. Indeed, consider any connected set $\Gamma \subset \mathbb{R}^n$, and fix $x, y \in \Gamma$. Then consider the map $\pi_x(z) = |z - x|$. It is clear that π_x is 1-Lipschitz, and from the connectedness of Γ it follows easily that $\pi_x(\Gamma) \supset [0, |x - y|]$. Hence,

$$|x - y| = \mathcal{H}^1([0, |x - y|]) \leq \mathcal{H}^1(\pi_x(\Gamma)) \leq \mathcal{H}^1(\Gamma),$$

and now the inequality $\operatorname{diam}(\Gamma) \leq \mathcal{H}^1(\Gamma)$ follows by taking a $\sup$ on the left hand side.

Proof of Lemma 3.5. If $\operatorname{diam}(T) \leq \delta$, there is nothing to prove. So, assume $\operatorname{diam}_\gamma(T) \geq \operatorname{diam}(T) > \delta$ (the first inequality follows by an argument similar to the one in Remark 3.6). Fix any point $y_0 \in T$, and let $M_0 := \sup\{d_\gamma(z, y) : z \in T\}$. Note that

$$\frac{\delta}{2} < M_0 \leq \operatorname{diam}_\gamma(T),$$

because otherwise $\operatorname{diam}_\gamma(T) \leq 2M_0 \leq \delta$. Fix any point $z_0 \in T$ with $d_\gamma(y_0, z_0) \geq M_0 - \frac{\delta}{6}$, and finally fix $x \in \gamma_{y_0,z_0}$ with $d_\gamma(y_0, x) = M_0 - \frac{\delta}{2}$.

Next, consider the equivalence relation $\sim$ on $T \setminus \{x\}$:

$$v \sim w \quad \Longleftrightarrow \quad x \notin \gamma_{v,w}.$$

The equivalence class of $v \in T \setminus \{x\}$ is denoted by $[v]$. Let

$$T^x := \{x\} \cup \{v \in T \setminus \{x\} : v \not\sim y\}.$$

Then T^x is a tree. To see this, consider a pair of points $v, w \in T^x$. Then $\gamma_{x,v} \setminus \{x\} \subset [v]$ and $\gamma_{x,w} \setminus \{x\} \subset [w]$, because clearly every pair of points in $\gamma_{x,v} \setminus \{x\}$ (resp. $\gamma_{x,w} \setminus \{w\}$) can be joined to v (resp. w) without passing through x. It follows that

$$\gamma_{v,x} \cup \{x\} \cup \gamma_{x,w} \subset [v] \cup \{x\} \cup [w] \subset T^x,$$

which implies that T^x is path connected, hence a tree (uniqueness is inherited from T).

I claim that $\operatorname{diam}(T^x) \leq \operatorname{diam}_\gamma(T^x) \leq \delta$. To see this, note that if $v \in T^x$, then $v \not\sim y_0$, which implies that $x \in \gamma_{v,y_0}$, and consequently

$$M_0 \geq d_\gamma(v, y_0) = d_\gamma(y_0, x) + d_\gamma(x, v) = (M_0 - \tfrac{\delta}{2}) + d_\gamma(x, v).$$

This gives $d_\gamma(x, v) \leq \frac{\delta}{2}$, and so $\operatorname{diam}_\gamma(T_x) \leq \delta$. Since obviously $\operatorname{diam}_\gamma(T_x) \leq \mathcal{H}^1(T_x)$, we see that T_x is a tree satisfying (a). Furthermore, $\mathcal{H}^1(T^x) \geq \frac{\delta}{3}$. Indeed, since $x \in \gamma_{y_0,z_0}$, one has $z_0 \not\sim y_0$, and consequently $\gamma_{x,z_0} \subset T^x$. This implies that

$$\mathcal{H}^1(T^x) \geq d_\gamma(x, z_0) \geq d_\gamma(y_0, z_0) - d_\gamma(y_0, x) \geq (M_0 - \tfrac{\delta}{2}) - (M_0 - \tfrac{\delta}{6}) = \frac{\delta}{3}. \tag{3.7}$$

Finally, observing that $(T \setminus T^x) \cup \{x\} = [y_0] \cup \{x\}$ is also a tree, one just needs to iterate the construction, chopping off another tree from $T \setminus T_x$. Since the part removed always has measure $\geq \frac{\delta}{3}$ by (3.7), the number of iterations is bounded by $\lesssim \mathcal{H}^1(T)/\delta + 1$, as claimed in (b). The proof is complete. □

Now, for the main result of the section:

Theorem 3.8 (Lower semicontinuity of length of continuums)**.** *Let $\{\Gamma_k\}_{k\in\mathbb{N}}$ be a sequence of compact continua in $\mathbb{R}^n$, convergent in the Hausdorff metric to a compact space Γ. Then Γ is a continuum, and*

$$\mathcal{H}^1(\Gamma) \leq \liminf_{k\to\infty} \mathcal{H}^1(\Gamma_k). \tag{3.9}$$

Proof. If Γ were not connected, there would be open disjoint sets U_1, U_2 with $\Gamma \subset U_1 \cup U_2$. Then, it follows from compactness, and the definition of Hausdorff convergence, that $\Gamma_k \subset U_1 \cup U_2$ for sufficiently large k, a contradiction.

To prove (3.9), one may assume that $\mathcal{H}^1(\Gamma_k) \leq C < \infty$ for all $k \in \mathbb{N}$, and also that the full sequence of numbers $\mathcal{H}^1(\Gamma_k)$ converges to the value on the right hand side of (3.9), say $L \leq C$ (because the following considerations are valid for any subsequence). For each $k \in \mathbb{N}$, choose a finite subset $S_k \subset \Gamma_k$ so that

$$d_H(S_k, \Gamma) \to 0$$

as $k \to \infty$. Since Γ_k is arcwise connected, there exist trees T_k such that $S_k \subset T_k \subset \Gamma_k$. To see this, declare any singleton $\{s_0\} \subset S_k$ as an initial tree T_k^0. Then, assume that T_k^j has been constructed for some j, consisting of finitely many arcs, and assume that at least one $s_{j+1} \in S_k$ yet lies outside T_k^j. Connect s_{j+1} to any point of T_k^j by an arc $\gamma = \gamma([0,1]) \subset \Gamma_k$, with $\gamma(0) = s_{j+1}$. Then there exists a smallest number $t \in (0,1]$ such that $\gamma(t) \in T_k^j$, and now $T_k^{j+1} := T_k^j \cup \gamma([0,t])$ is a tree containing s_{j+1}.

It is clear that

$$d_H(T_k, \Gamma) \to 0$$

as $k \to \infty$. Fix $\delta > 0$, and decompose every tree T_k as a $\mathcal{H}^1$-essentially disjoint union

$$T_k = \bigcup_{j=1}^{m_k} T_{k,j},$$

where $\operatorname{diam}(T_{k,j}) \leq \min\{\delta, \mathcal{H}^1(T_{k,j})\}$ and $m_k \lesssim C/\delta + 1$. Without loss of generality, one may assume that $m_k = m$ for all k (there are only finitely many choices for m_k, so this is anyway true after passing to a subsequence).

By the Blaschke selection theorem, every sequence $\{T_{k,j}\}_{k\in\mathbb{N}}$, $1 \leq j \leq m$, has a subsequence convergent in the Hausdorff metric to a non-empty compact continuum $\Gamma^j \subset \Gamma$; by re-indexing appropriately, and possibly finding subsequences inside subsequences, one may assume that the full sequences converge. It is clear that $\operatorname{diam}(\Gamma^j) \leq \delta$ and $\Gamma \subset \bigcup \Gamma^j$. It follows from the definition of $\mathcal{H}^1_\delta$, and the $\mathcal{H}^1$ essential disjointness of the trees $T_{k,j}$, $1 \leq j \leq m$, that

$$\mathcal{H}^1_\delta(\Gamma) \leq \sum_{j=1}^m \operatorname{diam}(\Gamma_j) = \limsup_{k\to\infty} \sum_{j=1}^m \operatorname{diam}(T_{k,j}) \leq \limsup_{k\to\infty} \sum_{j=1}^k \mathcal{H}^1(T_{k,j}) \leq \lim_{k\to\infty} \mathcal{H}^1(\Gamma_k) = L.$$

Letting $\delta \to 0$ proves the theorem. □

Remark 3.10. Note how the connectedness of the trees $T_{k,j}$ was used int the second-to-last inequality. This estimate would not work, if the sets Γ_j were chopped up into smaller pieces with some dumber procedure.

The following corollary will be useful in the sequel:

Corollary 3.11 (Existence of a shortest covering continuum)**.** *Assume that $K \subset \mathbb{R}^n$ is a compact set, and there exists a continuum $\Gamma_0 \supset K$ of finite length. Then, there exists a continuum $\Gamma \supset K$ of finite and minimal length.*

Proof. Choose a sequence of continuums $(\Gamma_k)_{k\in\mathbb{N}}$ with $\Gamma_k \supset E$, and such that

$$\mathcal{H}^1(\Gamma_k) \to \inf\{\mathcal{H}^1(\Gamma) : \Gamma \text{ is a continuum with } \Gamma \supset K\} =: m_K < \infty.$$

One may clearly assume that all the continuums are contained in a sufficiently large ball containing K, so the Blaschke selection theorem is applicable: there exists a subsequence $(k_j)_{j\in\mathbb{N}}$ and a compact set Γ with $d_H(\Gamma_{k_j}, \Gamma) \to 0$. It is now easy to check that $K \subset \Gamma$. By Theorem 3.8, moreover, the set Γ is a continuum, and satisfies

$$m_K \leq \mathcal{H}^1(\Gamma) \leq \liminf_{k\to\infty} \mathcal{H}^1(\Gamma_k) = m_K.$$

This proves the corollary. □

4. The L^∞ traveling salesman theorem of P. Jones

This section contains the proof of Peter Jones' original traveling salesman theorem for the numbers β_∞, but only in the plane. Recall the statement:

Theorem 4.1 (Jones)**.** *Let $\mathcal{D}$ be family of closed dyadic cubes in $\mathbb{R}^n$, and let $E \subset \mathbb{R}^n$ be a compact set satisfying*

$$\beta_\infty^2(E) := \sum_{\substack{Q\in\mathcal{D}_E \\ Q\subset 3E}} \beta_{E,\infty}^2(2Q)\ell(Q) < \infty,$$

where $\mathcal{D}_E = \{Q \in \mathcal{D} : Q \cap E \neq \emptyset\}$. Then, for any $\delta > 0$, there exists a compact connected set $\Gamma \subset \mathbb{R}^n$ such that $E \subset \Gamma$, and

$$\mathcal{H}^1(\Gamma) \leq (1+\delta)\operatorname{diam}(E) + C_\delta \beta_\infty^2(E). \tag{4.2}$$

Exercise 4.3. Is it possible to eliminate the $\delta > 0$ altogether?

Definition 4.4 (Convex hulls and extreme points)**.** The *convex hull* of a bounded set $K \subset \mathbb{R}^2$, denoted by $\operatorname{conv}(K)$, is the minimal (relative to inclusion) convex set $R \subset \mathbb{R}^2$ with $K \subset R$. Such a set exists, and in fact

$$\operatorname{conv}(K) = \bigcap_{\substack{K\subset R \\ R \text{ convex}}} R.$$

It is useful to note that $\operatorname{conv}(K)$ can also be expressed as the set of all (finite) convex combinations of points in K. That is,

$$\operatorname{conv}(K) = \left\{ \sum_{k=1}^m \lambda_k x_k : m \in \mathbb{N},\ \lambda_j \in [0,1],\ x_j \in K \text{ and } \sum_{k=1}^m \lambda_j = 1 \right\}. \tag{4.5}$$

To see this, simply note that the set on the left hand side is convex and contains K, and is clearly contained in any set with these properties.

A convex combination as on the right hand side of (4.5) is called *non-trivial*, if $\lambda_j < 1$ for all $1 \leq j \leq m$. The set of *extreme points* of $K \subset \mathbb{R}^2$, denoted by $\operatorname{Ex}(K)$, are those points in K, which cannot be expressed as non-trivial convex combinations of elements in K. In other words $x \in \operatorname{Ex}(K)$, if and only if $x \in K$, and the following holds: if x has a representation

$$x = \sum_{k=1}^m \lambda_k x_k, \qquad x_j \in K,$$

with $0 \leq \lambda_j \leq 1$ and $\sum \lambda_k = 1$, then $\lambda_j = 0$ for all but one index $j = j_0$, and $x_{j_0} = x$.

It is slightly, but not very, non-trivial but true that the convex hull of a compact set is a compact set:

Lemma 4.6. *If $K \subset \mathbb{R}^2$ is compact, then $\operatorname{conv}(K)$ is compact.*

Proof. The details are contained in [16, Theorem 3.25]. The main idea is the following: if $K \subset \mathbb{R}^2$ is bounded, then $\operatorname{conv}(K)$ can be expressed as

$$\operatorname{conv}(K) = f(S \times K \times K \times K), \tag{4.7}$$

where $S \subset \mathbb{R}^3$ is the (compact) simplex $S = \{(\lambda_1, \lambda_2, \lambda_3) : \lambda_i \in [0,1] \text{ and } \sum \lambda_i = 1\}$, and

$$f(\lambda_1, \lambda_2, \lambda_3, x_1, x_2, x_3) = \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3.$$

Since f is continuous and K is assumed compact, the right hand side of (4.7) is clearly compact, and it suffices to prove (4.7). This takes a bit of linear algebra: if x is any convex combination of m points in K, then it is actually the convex combination of **three** points in K. In $\mathbb{R}^n$ this the same is generally true with "three" replaced by "$(n+1)$" (so the lemma remains valid in $\mathbb{R}^n$). For the remaining details, see [16]. □

Lemma 4.8. $\operatorname{Ex}(\operatorname{conv}(K)) \subset K$ *for all bounded sets* $K \subset \mathbb{R}^2$.

Proof. If $x \in \operatorname{Ex}(\operatorname{conv}(K))$, then $x \in \operatorname{conv}(K)$ by definition of "Ex", and hence x can be represented as a convex combination of points in $K \subset \operatorname{conv}(K)$. If $x \notin K$, then the combination is necessarily non-trivial, and hence $x \notin \operatorname{Ex}(\operatorname{conv}(K))$ contrary to assumption. □

Now, we start the proof of Theorem 4.1. The argument is copied nearly verbatim from the book of Bishop and Peres, see [2, Theorem 10.5.1]. For a closed convex set $R \subset \mathbb{R}^2$, define the following variant of the β_∞-number:

$$\beta(R) := \max_L \sup_{x \in R} \frac{\operatorname{dist}(x, L)}{\operatorname{diam}(R)},$$

where the "$\max_L$" is taken over all *chords* of R of length $\mathcal{H}^1(L) = \operatorname{diam}(R)$. A *chord* is a line segment with both endpoints on ∂R. A simple compactness argument shows that the first $\max$ is well-defined, and attained for some chord L_R, and L_R will be called *the diameter* of R. Note that $\operatorname{dist}(z, L_R) \leq \beta(R) \operatorname{diam}(R)$ for all $x \in R$, whence R can be covered by a rectangle of dimensions $\operatorname{diam}(R) \times 2\beta(R)$ parallel to L_R.

4.1. **Construction and connectedness.** The curve Γ, covering E, will be obtained as the intersection of a nested sequence of compact connected sets Γ_n, each containing E. Every set Γ_n has the form

$$\Gamma_n = \bigcup_{R \in \mathcal{R}_n} R \cup \bigcup_{k=0}^{n} \bigcup_{B \in \mathcal{B}_k} B,$$

where the family $\mathcal{R}_n$ consists of interior-disjoint closed convex sets $\mathcal{R}_n$, and the families $\mathcal{B}_k$ consist of closed line segments, called *bridges*. Note that bridges are never deleted: Γ_{n+1} contains all the bridges contained in Γ_n.

The following properties will be maintained throughout the construction, and they will guarantee that each set Γ_n is connected:

Property 1 (Connectivity)**.** For $n \in \mathbb{N}$, let

$$\mathcal{F}_n := \mathcal{R}_n \cup \bigcup_{k=0}^{n} \mathcal{B}_k$$

be the family of sets such that $\Gamma_n = \cup\{F : F \in \mathcal{F}_n\}$. First, the extreme points $\mathrm{Ex}(R)$ of every set $R \in F_n$ lie in E. Second, if $K_1, K_2 \in \mathcal{F}_n$, then then K_1 and K_2 can be joined by an *extreme point tour*: there exist sets

$$K_1 = E_1, E_2, \ldots, E_{m-1}, E_m = K_2 \in \mathcal{F}_n$$

such that $\mathrm{Ex}(E_j) \cap \mathrm{Ex}(E_{j+1}) \neq \emptyset$ for $1 \leq j \leq m-1$. In particular, Γ_n is connected.

Now, the construction begins. Set

$$\mathcal{R}_0 := \{\mathrm{conv}(E)\} \quad \text{and} \quad \mathcal{B}_0 = \emptyset.$$

Then Property 1 is satisfied by Lemma 4.8. Assume that $\mathcal{R}_n, \mathcal{B}_n$ have already been defined for some $n \geq 0$. Then, $\mathcal{R}_{n+1}$ will be defined by replacing every set $R \in \mathcal{R}_n$ by two further interior-disjoint closed convex sets, called the *children* of R. The children of R may be connected by a line segment, which is added to $\mathcal{B}_{n+1}$. In particular, no sets are ever deleted "later" from $\mathcal{B}_n$.

Fix $R \in \mathcal{R}_n$. The construction now divides to two cases.

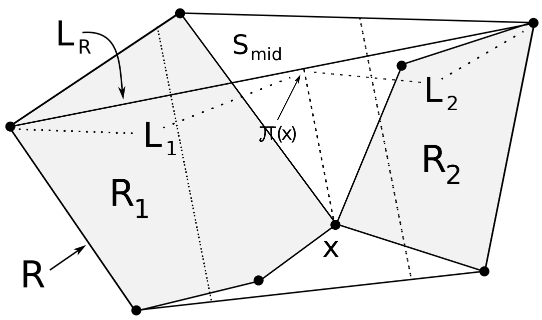

FIGURE 2. The case, where no bridge is added to $\mathcal{B}_{n+1}$.

Case (NB)**.** Here "NB" stands for "no bridge". Let S_{mid} be the closed middle third of the diameter chord L_R, see Figure 2. Let π be the orthogonal projection to L_R. In the case (NB), assume that

$$E \cap R \cap \pi^{-1}(S_{\mathrm{mid}}) \neq \emptyset,$$

and pick any point $x \in E \cap R \cap \pi^{-1}(S_{\mathrm{mid}})$. Then, divide L_R into two closed sub-segments L_1 and L_2, with a common endpoint at $\pi(x)$. Define R_1 and R_2 to be the convex hulls of the sets

$$E \cap R \cap \pi^{-1}(L_1) \quad \text{and} \quad E \cap R \cap \pi^{-1}(L_2),$$

respectively. It is easy to see that that $\pi^{-1}(\pi(x))$ contains a point in $\mathrm{Ex}(R_1) \cap \mathrm{Ex}(R_2)$, so in particular

$$\mathrm{Ex}(R_1) \cap \mathrm{Ex}(R_2) \neq \emptyset.$$

In fact, the set $R_1 \cap R_2$ is a (possibly degenerate) line segment, whose endpoints lie in $\mathrm{Ex}(R_1) \cap \mathrm{Ex}(R_2)$.

To check that Property 1 remains valid, first note that $\mathrm{Ex}(R_1), \mathrm{Ex}(R_2) \subset E$: indeed, R_1, R_2 are the convex hulls of certain compact subsets of E, and one can just apply Lemma 4.8. So, it remains to check that the "extreme points tour" property remains valid. To this end, first note that

$$\mathrm{Ex}(R) \subset \mathrm{Ex}(R_1) \cup \mathrm{Ex}(R_2). \tag{4.9}$$

This is because if $x \in \mathrm{Ex}(R)$, then $x \in E$ by Property 1, and so evidently $x \in R_1 \cup R_1$. Assume, for instance, that $x \in R_1$. Now, if x could be expressed as a non-trivial convex combination of elements in R_1, then it could certainly be expressed as such a combination of elements in R, which would violate $x \in \mathrm{Ex}(R)$. It follows that $x \in \mathrm{Ex}(R_1)$, which proves (4.9).

Finally, let $K_1, K_2 \in \mathcal{F}_{n+1}$ be sets as in Property 1. The task is to find an extreme point tour in $\mathcal{F}_{n+1}$, connecting K_1 to K_2. For $j \in \{1,2\}$, write

$$\hat{K}_j := \begin{cases} K_j, & \text{if } K_j \notin \{R_1, R_2\}, \\ R, & \text{if } K_j \in R_1, R_2. \end{cases}$$

Then $\hat{K}_1, \hat{K}_2 \in \mathcal{F}_n$, and there exists an extreme point tour

$$\hat{K}_1 = E_1, E_2, \ldots, E_{m-1}, E_m = \hat{K}_2 \in \mathcal{F}_n$$

If $E_j \neq R$ for all $1 \leq j \leq m$, then $E_1, \ldots, E_m$ is also an extreme point tour in $\mathcal{F}_{n+1}$, connecting $\hat{K}_1$ to $\hat{K}_2$. Otherwise, if $E_j = R$ for some $1 \leq j \leq m$, the tour does not lie in $\mathcal{F}_{n+1}$, and one needs to modify it. Assume first that $1 < j < m$, and $E_j = R$. By definition of the tour, $\mathrm{Ex}(E_{j-1}) \cap \mathrm{Ex}(E_j) \neq \emptyset$ and $\mathrm{Ex}(E_j) \cap \mathrm{Ex}(E_{j+1}) \neq \emptyset$. Since $\mathrm{Ex}(E_j) = \mathrm{Ex}(R) \subset \mathrm{Ex}(R_1) \cup \mathrm{Ex}(R_2)$ by (4.9), one has either

$$\mathrm{Ex}(E_{j-1}) \cap \mathrm{Ex}(R_1) \neq \emptyset \quad \text{or} \quad \mathrm{Ex}(E_{j-1}) \cap \mathrm{Ex}(R_2) \neq \emptyset.$$

For instance, assume that $\mathrm{Ex}(E_{j-1}) \cap \mathrm{Ex}(R_1) \neq \emptyset$. Similarly, either

$$\mathrm{Ex}(E_{j+1}) \cap \mathrm{Ex}(R_1) \neq \emptyset \quad \text{or} \quad \mathrm{Ex}(E_{j+1}) \cap \mathrm{Ex}(R_2) \neq \emptyset.$$

Assume for instance that $\mathrm{Ex}(E_{j+1}) \cap \mathrm{Ex}(R_2) \neq \emptyset$. Because also $\mathrm{Ex}(R_1) \cap \mathrm{Ex}(R_2) \neq \emptyset$, the set $E_j = R$ can be replaced by $E_j^1 = R_1$ and $E_j^2 = R_2$, and $E_1, \ldots, E_j^1, E_j^2, \ldots, E_m$ remains an an extreme point tour connecting $\hat{K}_1$ to $\hat{K}_2$. Once all occurrences of $E_j = R$, $1 < j < m$, have been replaced in this manner, then we have a tour connecting $\hat{K}_1$ to $\hat{K}_2$ in $\mathcal{F}_{n+1}$, apart possibly from the endpoints. If one of the endpoints does not lie in $\mathcal{F}_{n+1}$, say $\hat{K}_1 \notin \mathcal{F}_{n+1}$, this means precisely that $\hat{K}_1 = R$, and $K_1 \in \{R_1, R_2\}$. Then, repeating the argument from above, $E_1 = \hat{K}_1 = R$ can be replaced by either, or both of, the sets R_1, R_2, while maintaining the extreme point tour property. This gives a tour connecting K_1 to K_2 in $\mathcal{F}_{n+1}$, as desired.

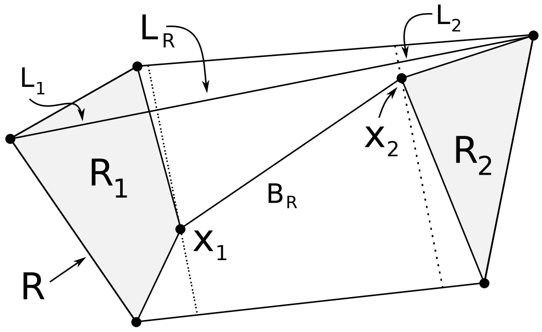

FIGURE 3. The case, where a new bridge B_R is added to $\mathcal{B}_{n+1}$.

Case (B)**.** Here "B" stands for "bridge". Recall the notation from the previous case, and this time assume that

$$E \cap R \cap \pi^{-1}(S_{\mathrm{mid}}) = \emptyset.$$

Thus, if $L_R = S_{\mathrm{left}} \cup S_{\mathrm{mid}} \cup S_{\mathrm{right}}$, then

$$E \cap R \subset \pi^{-1}(S_{\mathrm{left}}) \cup \pi^{-1}(S_{\mathrm{right}}).$$

It follows that there are two minimal intervals $L_1 \subset S_{\mathrm{left}}$ and $L_2 \subset S_{\mathrm{right}}$ such that

$$E \cap R \subset \pi^{-1}(L_1) \cup \pi^{-1}(L_2),$$

see Figure 3. As in the previous case, define $R_1 := \mathrm{conv}[E \cap R \cap \pi^{-1}(L_1)]$ and $R_2 := \mathrm{conv}[E \cap R \cap \pi^{-1}(L_2)]$. Property 1 remains valid by similar considerations as in the previous case.

Now, the construction of the sets Γ_n is complete, and Γ is defined by

$$\Gamma := \bigcap_{n \geq 0} \Gamma_n.$$

It is clear that $\Gamma \supset E$. Also, Γ is connected: if $\Gamma \subset U_1 \cup U_2$ with U_1, U_2 disjoint open sets, then $\Gamma_n \subset U_1 \cup U_2$ for sufficiently large n (otherwise $(\Gamma_n \setminus [U_1 \cup U_2])_{n \in \mathbb{N}}$ would be a sequence of nested non-empty compact sets with empty intersection). But this is impossible, since Γ_n is connected for all n.

4.2. **Length estimates.** It remains to prove the length bound (4.2). To this end, the following inequality will be first verified:

$$\sum_{R \in \mathcal{R}_n} \mathrm{diam}(R) + \sum_{k=0}^{n} \sum_{B \in \mathcal{B}_k} \mathcal{H}^1(B) \leq (1+\delta)\,\mathrm{diam}(E) + C_\delta \sum_{k=0}^{n-1} \sum_{R \in \mathcal{R}_n} \beta^2(R)\,\mathrm{diam}(R). \quad (4.10)$$

where $\delta > 0$, and $C_\delta \geq 1$ only depends on δ. For line segments, such as $B \in \mathcal{B}_n$ or L_R, I will abbreviate $\mathcal{H}^1(\cdot) =: |\cdot|$ in the sequel.

Lemma 4.11. *Assume that $R \in \mathcal{R}_n$ is replaced by $R_1, R_2 \in \mathcal{R}_{n+1}$ and $B_R \in \mathcal{B}_{n+1}$. Then*

$$\mathrm{diam}(R_1) + \mathrm{diam}(R_2) + \frac{|B_R|}{1+\delta} \leq \mathrm{diam}(R) + C_\delta \beta^2(R)\,\mathrm{diam}(R).$$

Proof. Consider Case (NB) first, and recall the two line segments $L_1, L_2 \subset L_R$ with common endpoint $\pi(x)$. Note that

$$\min\{|L_1|, |L_2|\} \geq \frac{|L_R|}{3} = \frac{\mathrm{diam}(R)}{3}. \quad (4.12)$$

Since R_j is contained in a rectangle with dimensions $|L_j| \times 2\beta(R)\,\mathrm{diam}(R)$ parallel to L_R, one has, using (4.12), that[5]

$$\mathrm{diam}(R_j) \leq \sqrt{|L_j|^2 + 4\beta^2(R)\,\mathrm{diam}(R)^2} \leq |L_j| + 6\beta^2(R)\,\mathrm{diam}(R). \quad (4.13)$$

This should be contrasted with the "trivial estimate"

$$\mathrm{diam}(R_j) \leq \sqrt{|L_j|^2 + 4\beta^2(R)\,\mathrm{diam}(R)^2} \leq |L_j| + 2\beta(R)\,\mathrm{diam}(R), \quad (4.14)$$

[5]You may first wish to check, **ab**stractly, that $a \geq c/3$ implies $\sqrt{a^2 + 4bc^2} \leq a + 6bc$ for $a, b, c \geq 0$.

which holds without any assumptions on $|L_j|$. Since $|L_1| + |L_2| \leq \operatorname{diam}(R)$, the claim of the lemma then follows from (4.13) with $C_\delta = 12$.

Next, consider Case (B). Let $J_R \subset L_R$ be the segment $J_R = L_R \setminus (L_1 \cup L_2)$, where L_1 and L_2 are defined as in Case (B). Then

$$|B_R| \leq \sqrt{|J_R|^2 + 4\beta^2(R)\operatorname{diam}(R)^2} \leq |J_R| + 2\beta(R)\operatorname{diam}(R). \tag{4.15}$$

Now, there are two subcases. First, if $\beta(R) \geq \theta := \delta/100$, then

$$\operatorname{diam}(R_1) + \operatorname{diam}(R_2) + \frac{|B_R|}{1+\delta} \leq 3\operatorname{diam}(R) \leq \operatorname{diam}(R) + \left(\frac{2}{\theta^2}\right)\beta^2(R)\operatorname{diam}(R).$$

If $\beta(R) < \theta$, then, combining (4.15) with the "trivial estimate" (4.13) gives

$$\begin{aligned}\operatorname{diam}(R_1) + \operatorname{diam}(R_2) + \frac{|B_R|}{1+\delta} &\leq |L_1| + |L_2| + 4\theta\operatorname{diam}(R) + \frac{|J_R| + 2\theta\operatorname{diam}(R)}{1+\delta}\\ &= \frac{1 + 2\delta/3 + 4\theta(1+\delta) + 2\theta}{1+\delta}\operatorname{diam}(R),\end{aligned} \tag{4.16}$$

noting that $|L_1|+|L_2|+|J_R| = \operatorname{diam}(R)$, and $|L_1|+|L_2| \leq (2/3)\operatorname{diam}(R)$. Since $\theta = \delta/100$, the factor of $\operatorname{diam}(R)$ on line (4.16) is ≤ 1, and the lemma follows. □

Now the time is ripe to prove (4.10). For $R \in \mathcal{R}_n$, write $\operatorname{ch}(R) := \{R_1, R_2\} \subset \mathcal{R}_{n+1}$ for the children of R, and write $B_R \in \mathcal{B}_{n+1}$ for the bridge associated to R; in Case (NB), let $B_R := \emptyset$. Then, for any $n \geq 1$, using Lemma 4.11,

$$\begin{aligned}&\sum_{R\in\mathcal{R}_n}\operatorname{diam}(R) + \frac{1}{1+\delta}\sum_{k=0}^{n}\sum_{B\in\mathcal{B}_k}|B|\\ &= \sum_{R\in\mathcal{R}_{n-1}}\left[\sum_{R'\in\operatorname{ch}(R)}\operatorname{diam}(R') + \frac{|B_R|}{1+\delta}\right] + \frac{1}{1+\delta}\sum_{k=0}^{n-1}\sum_{B\in\mathcal{B}_k}|B|\\ &\overset{\text{L.4.11}}{\leq} \left[\sum_{R\in\mathcal{R}_{n-1}}\operatorname{diam}(R) + \frac{1}{1+\delta}\sum_{k=0}^{n-1}\sum_{B\in\mathcal{B}_k}|B|\right] + C_\delta\sum_{R\in\mathcal{R}_{n-1}}\beta^2(R)\operatorname{diam}(R).\end{aligned}$$

The term in brackets on the last line is of the same form as the term on the first line, so the estimate can be iterated, and n repetitions give

$$\sum_{R\in\mathcal{R}_n}\operatorname{diam}(R) + \frac{1}{1+\delta}\sum_{k=0}^{n}\sum_{B\in\mathcal{B}_k}|B| \leq \operatorname{diam}(R_0) + C_\delta\sum_{k=0}^{n-1}\sum_{R\in\mathcal{R}_k}\beta^2(R)\operatorname{diam}(R).$$

Multiplying both sides by $(1+\delta)$ gives (4.10), recalling that $\operatorname{diam}(R_0) = \operatorname{diam}(E)$.

The next task will be to prove

$$\sum_{k=0}^{\infty}\sum_{R\in\mathcal{R}_k}\beta^2(R)\operatorname{diam}(R) \lesssim \beta_\infty^2(E). \tag{4.17}$$

Let's start with an easy observation: for any $R \in \mathcal{R}_n$,

$$\max\{\mathcal{L}^2(R_1), \mathcal{L}^2(R_2)\} \leq \tau\mathcal{L}^2(R) \tag{4.18}$$

for some absolute constant $\tau < 1$. To see this, let $\Delta \subset R$ be the triangle spanned by the endpoints of L_R, and some point $x \in \partial R$ with $\operatorname{dist}(x, L_R) = \beta(R)\operatorname{diam}(R)$. Then

$$\mathcal{L}^2(R) \sim \mathcal{L}^2(\Delta) = \frac{\beta(R)\operatorname{diam}(R)^2}{2} \tag{4.19}$$

and

$$\mathcal{L}^2(\Delta \cap \pi^{-1}(S_{\text{left}})) \sim \mathcal{L}^2(\Delta) \sim \mathcal{L}^2(\Delta \cap \pi^{-1}(S_{\text{right}}))$$

with absolute constants. It follows that

$$\mathcal{L}^2(R_1) \leq \mathcal{L}^2(R) - \mathcal{L}^2(\Delta \cap \pi^{-1}(S_{\text{right}})) \leq \tau \mathcal{L}^2(R),$$

and a similar estimate holds for $\mathcal{L}^2(R_2)$. This proves (4.18). Next, a similar estimate is desired for the **diameters** of the convex sets:

Lemma 4.20. *There is an absolute constant $N \in \mathbb{N}$ with the following property. If $R \in \mathcal{R}_n$ and $R' \in \mathcal{R}_{n+N}$ with $R' \subset R$, then*

$$\operatorname{diam}(R') \leq \frac{3\operatorname{diam}(R)}{4}.$$

Proof. If the number $\beta(R)$ is small enough, say $\beta(R) \leq \theta$, then

$$\max\{\operatorname{diam}(R_1), \operatorname{diam}(R_2)\} \leq 3\operatorname{diam}(R)/4.$$

(Just have a look at the estimates for $\operatorname{diam}(R_1)$ within Lemma 4.11, if you are unsure.) So, if

$$R = R^{(0)} \supset R^{(1)} \supset \ldots \supset R^{(N)} = R'$$

is a sequence with $R^{(j)} \in \mathcal{R}_{n+j}$, and if $\beta(R^{(N-1)}) \leq \theta$, then

$$\operatorname{diam}(R') \leq 3\operatorname{diam}(R^{(N-1)})/4 \leq 3\operatorname{diam}(R)/4,$$

and the proof is complete. Otherwise $\beta(R^{(N-1)}) \geq \theta$, and by (4.18)-(4.19),

$$\theta \operatorname{diam}(R')^2 \leq \theta \operatorname{diam}(R^{(N-1)})^2 \lesssim \mathcal{L}^2(R^{(N-1)}) \leq \tau^{N-1}\mathcal{L}^2(R) \lesssim \tau^{N-1}\operatorname{diam}(R)^2.$$

This proves that, in every case, $\operatorname{diam}(R') \leq 3\operatorname{diam}(R)/2$, if N is large enough. □

Now, the table is set for (4.17). For a dyadic square Q, let

$$\mathcal{R}(Q) := \bigcup_{n \geq 0} \{R \in \mathcal{R}_n : R \cap Q \neq \emptyset \text{ and } \ell(Q)/2 < \operatorname{diam}(R) \leq \ell(Q)\}.$$

I claim that the collection $\mathcal{R}(Q)$ has bounded "interior overlap", that is,

$$\sum_{R \in \mathcal{R}(Q)} \chi_{\operatorname{int} R} \lesssim 1. \tag{4.21}$$

To see this, note that two sets $R_1 \in \mathcal{R}_m$ and $R_2 \in \mathcal{R}_n$, $m, n \geq 0$, can have shared interior, only if either R_1 is a descendant of R_2, or vice versa. So, if $x \in \operatorname{int} R_i$ for $R_i \in \mathcal{R}(Q)$, $1 \leq i \leq M$, then the sets $R_1, \ldots, R_M$ are nested. However, by Lemma 4.20, the diameters of the convex sets decay rapidly in long nested sequences; since $\ell(Q)/2 < \operatorname{diam}(R_i) \leq \ell(Q)$ for all $1 \leq i \leq M$, this sets an upper bound for M and proves (4.21).

Another observation about $\mathcal{R}(Q)$ is the following: if $R \in \mathcal{R}(Q)$, then $R \subset 2Q$, because $R \cap Q \neq \emptyset$ and $\operatorname{diam}(R) \leq \ell(Q)$. In particular, $E \cap R \subset E \cap 2Q$ is contained in $W \cap 2Q$, where W is a strip of width $2\beta_{E,\infty}(2Q)\ell(2Q)$. Since $W \cap 2Q$ is convex, it follows that

$$R = \operatorname{conv}(E \cap R) \subset W \cap 2Q, \qquad R \in \mathcal{R}(Q). \tag{4.22}$$

Next, use (4.19), (4.21) and (4.22) to make the following estimate:

$$\begin{aligned}\sum_{R \in \mathcal{R}(Q)} \beta(R)\,\mathrm{diam}(R)^2 &\sim \sum_{R \in \mathcal{R}(Q)} \mathcal{L}^2(\mathrm{int}\,R)\\ &= \int_{W \cap 2Q} \sum_{R \in \mathcal{R}(Q)} \chi_{\mathrm{int}\,R}\,d\mathcal{L}^2\\ &\lesssim \mathcal{L}^2(W \cap 2Q) \lesssim \beta_{E,\infty}(2Q)\ell(Q)^2.\end{aligned}$$

Since $\mathrm{diam}(R) \sim \ell(Q)$ for all $R \in \mathcal{R}(Q)$, one concludes that

$$\sum_{R \in \mathcal{R}(Q)} \beta^2(R) \leq \Big(\sum_{R \in \mathcal{R}(Q)} \beta(R)\Big)^2 \lesssim \beta_{E,\infty}^2(2Q).$$

Finally, observe that every convex set $R \in \mathcal{R}_n$, for any $n \geq 0$ is contained in at least one of the sets $\mathcal{R}(Q)$, $Q \in \mathcal{D}_E$, with $\ell(Q) \leq 2\,\mathrm{diam}(E)$ (because $E \cap R \neq \emptyset$, and any point $x \in E \cap R$ is contained in a dyadic square $Q \in \mathcal{D}_E$ with $\ell(Q)/2 < \mathrm{diam}(R) \leq \ell(Q)$). Consequently,

$$\sum_{k=0}^{n} \sum_{R \in \mathcal{R}_k} \beta^2(R)\,\mathrm{diam}(R) \lesssim \sum_{\substack{Q \in \mathcal{D}_E \\ \ell(Q) \leq 2\,\mathrm{diam}(E)}} \ell(Q) \sum_{R \in \mathcal{R}(Q)} \beta^2(R) \lesssim \sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset 3E}} \beta_{E,\infty}^2(2Q)\ell(Q) = \beta_\infty^2(E),$$

which proves (4.17).

Combining (4.10) and (4.17), it has now been established that

$$\sum_{R \in \mathcal{R}_n} \mathrm{diam}(R) + \sum_{k=0}^{n} \sum_{B \in \mathcal{B}_k} \mathcal{H}^1(B) \leq (1+\delta)\,\mathrm{diam}(E) + \beta_\infty^2(E)$$

uniformly for $n \geq 0$. Since

$$\Gamma \subset \Gamma_n = \bigcup_{R \in \mathcal{R}_n} R \cup \bigcup_{k=0}^{n} \bigcup_{B \in \mathcal{B}_k} B, \quad n \geq 0,$$

the desired bound for $\mathcal{H}^1(\Gamma)$ now follows from the very definition of $\mathcal{H}^1$ (and the fact that the diameters of the sets in $\mathcal{R}_n$ tend to zero uniformly, as $n \to \infty$).

5. The L^1-traveling salesman theorem

This section contains two proofs of the L^1-traveling salesman theorem for doubling measures, Theorem 2.11, which is reproduced as Theorem 5.2 below. The result first appeared in a paper [1] of M. Badger and R. Schul from 2016; there it was obtained via a new "geometric traveling salesman theorem", which is the main topic in Section 7. However, an observation of X. Tolsa from [17, Section 7] allows one to reduce Theorem 5.2 to Jones' L^∞ traveling salesman theorem, **if** one assume that the doubling measure μ is *smooth* in the following sense:

Definition 5.1 (Smooth measures)**.** A Radon measure μ on $\mathbb{R}^n$ is *smooth*, if it is doubling, and there is a constant $\theta > 0$ such that

$$\mu(B(x,\theta r)) \leq \frac{\mu(B(x,r))}{2}, \qquad x \in \mathrm{spt}\,\mu,\ 0 < r \leq \mathrm{diam}(\mathrm{spt}\,\mu).$$

The reduction to Jones' theorem gives a (fairly) short proof of Theorem 5.2 in the plane, assuming smoothness. These considerations constitute the first half of the section. The second half contains the proof of Theorem 5.2 in full generality, following the argument of Badger and Schul.

I now recall the result:

Theorem 5.2 (Badger-Schul)**.** *Let μ be a doubling measure on $\mathbb{R}^n$ with compact support $E = \operatorname{spt}\mu$, and assume that the numbers $\beta_{\mu,1}$ satisfy*

$$\beta_1^2(\mu) := \sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset \lambda E}} \beta_{\mu,1}^2(2Q)\ell(Q) < \infty,$$

where $\lambda \geq 1$ is a sufficiently large constant depending only on n, and, as always, $\mathcal{D}_E = \{Q \in \mathcal{D} : Q \cap E \neq \emptyset\}$. Then $\operatorname{spt}\mu$ can be covered by a continuum $\Gamma \subset \mathbb{R}^n$ with

$$\mathcal{H}^1(\Gamma) \lesssim_{D_\mu,n} \operatorname{diam}(E) + \beta_1^2(\mu).$$

If the $\beta_{\mu,1}$-numbers satisfy a Carleson condition, then Γ can be taken to be 1-Ahlfors-David regular, at least in the plane:

Theorem 5.3 (Badger-Schul, Carleson version)**.** *Same assumptions as in Theorem 5.2, except that $n = 2$, and the finiteness of $\beta_1^2(\mu)$ is replaced by the Carleson condition*

$$\sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset R}} \beta_{\mu,1}^2(2Q)\ell(Q) \lesssim \ell(R), \qquad R \in \mathcal{D}_E.$$

Then $E = \operatorname{spt}\mu$ can be covered by an AD regular continuum.

Remark 5.4. Note that, in both theorems above, using the $\beta_{\mu,1}$-numbers gives the strongest possible result (as contrasted to the numbers $\beta_{\mu,p}$ for any $1 \leq p \leq \infty$).

5.1. Tolsa's observation.

Here is the main result of this subsection: it allows us to transfer assumptions on the $\beta_{\mu,1}$-numbers to those on $\beta_{E,\infty}$-numbers.

Theorem 5.5 (Tolsa)**.** *Let μ smooth measure with $E = \operatorname{spt}\mu \subset \mathbb{R}^2$. Then*

$$\sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset R}} \beta_{E,\infty}^2(2Q)\ell(Q) \lesssim \sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset 2R}} \beta_{\mu,1}^2(3Q)\ell(Q)$$

for any square $R \in \mathcal{D}$ with $\ell(R) \leq 10\operatorname{diam}(E)$.

Key to the proof is the following geometric lemma:

Lemma 5.6. *Let μ be a smooth measure with $E := \operatorname{spt}\mu$, and let $Q \in \mathcal{D}_E$ with $\ell(Q) \leq 10\operatorname{diam}(E)$. Let ℓ_Q be a line with $\beta_{\mu,1}(3Q, \ell_Q) = \beta_{\mu,1}(3Q)$. Then, for any $x \in E \cap 2Q$,*

$$\operatorname{dist}(x, \ell_Q) \lesssim \sum_{\substack{P \in \mathcal{D}_E \\ x \in P \subset 2Q}} \beta_{\mu,1}(3P)\ell(P),$$

where the implicit constants only depend on the doubling constant of μ.

The proof of the lemma needs two further lemmas. The first is an observation of G. Lerman, while the second is folklore.

Lemma 5.7. *Let $n \geq 2$ and $1 \leq p < \infty$. Let μ be a Radon measure, and let $B \subset \mathbb{R}^n$ be a set with $0 < \operatorname{diam}(B) < \infty$ and $0 < \mu(B) < \infty$. Let c_B be the* centre of mass *of μ in B, namely*

$$c_B := \frac{1}{\mu(B)} \int_B x \, d\mu x.$$

Then

$$\operatorname{dist}(c_B, \ell) \leq \beta_{\mu,p}(B, \ell) \operatorname{diam}(B)$$

for every straight line $\ell \subset \mathbb{R}^n$.

Proof. Fix a straight line $\ell \subset \mathbb{R}^n$ and note that the function $x \mapsto \operatorname{dist}(x, \ell)^p$ is convex for $p \in [1, \infty)$. So, by a vector-valued version of Jensen's inequality, see lemma below,

$$\operatorname{dist}(c_B, \ell)^p := \operatorname{dist}\left(\frac{1}{\mu(B)} \int_B x \, d\mu x, \ell\right)^p \leq \frac{1}{\mu(B)} \int_B \operatorname{dist}(x, \ell)^p \, d\mu x = \operatorname{diam}(B)^p \beta_{\mu,p}(B, \ell)^p.$$

□

Lemma 5.8 (Jensen's inequality). *Let $(X_1, \ldots, X_n)$ be a random vector on some probability space $(\Omega, \mathbb{P})$, and let $\varphi \colon \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then,*

$$\varphi(\mathbb{E}[(X_1, \ldots, X_n)]) \leq \mathbb{E}[\varphi(X_1, \ldots, X_n)].$$

Proof. We cheat by assuming that φ is differentiable everywhere, because this suffices for the application in Lemma 5.7 (at least for $p > 1$, and the case $p = 1$ can be obtained by taking limits). A consequence of convexity is the following inequality:

$$\varphi(x) - \varphi(y) \geq \nabla \varphi(y) \cdot (x - y), \qquad x, y \in \mathbb{R}^n. \tag{5.9}$$

Indeed, writing $v := (x - y)/|x - y|$, we have

$$\begin{aligned}
\nabla \varphi(y) \cdot (x - y) = [\nabla \varphi(y) \cdot v] \cdot |x - y| &= \left[\lim_{h \to 0+} \frac{\varphi(y + hv) - \varphi(y)}{h}\right] \cdot |x - y| \\
&= \left[\lim_{h \to 0+} \frac{\varphi((h/|x-y|)x + (1 - h/|x-y|)y) - \varphi(y)}{h}\right] \cdot |x - y| \\
&\leq \left[\limsup_{h \to 0+} \frac{(h/|x-y|)\varphi(x) + (1 - h/|x-y|)\varphi(y) - \varphi(y)}{h}\right] \cdot |x - y| \\
&= \varphi(x) - \varphi(y),
\end{aligned}$$

using convexity in the inequality.

In particular, setting $x = \bar{X} := (X_1, \ldots, X_n)$ and $y = \mathbb{E}[\bar{X}]$ in (5.9), and taking expectations on both sides,

$$\mathbb{E}[\varphi(\bar{X})] - \varphi(\mathbb{E}[\bar{X}]) \geq \mathbb{E}[\nabla \varphi(\mathbb{E}[\bar{X}]) \cdot (\bar{X} - \mathbb{E}[\bar{X}])] = 0.$$

This proves the lemma. □

Lemma 5.10. *Let μ be a smooth measure, and let P, R be (non-dyadic) cubes with $\ell(P) \sim \ell(R) \lesssim \operatorname{diam}(\operatorname{spt} \mu)$, and let ℓ_P, ℓ_R be lines, which minimise $\beta_{\mu,1}(P)$ and $\beta_{\mu,1}(R)$, respectively. Assume that*

$$\tau P \cap \tau R$$

contains a point of $\operatorname{spt} \mu$ for some $\tau < 1$. Then, ℓ_P and ℓ_R are very close in the following sense:

$$\operatorname{dist}(z, \ell_R) \lesssim_\tau \min\{\beta_{\mu,1}(P)\ell(P), \beta_{\mu,1}(R)\ell(R)\}, \quad z \in \ell_P \cap P.$$

By symmetry, the same holds for $\operatorname{dist}(z, \ell_P)$, for $z \in \ell_R \cap R$.

Proof. Start by finding two points $x_0, y_0 \in P \cap R$ with $|x_0 - y_0| \sim_\tau \ell(P) \sim \ell(R)$ with the property that

$$\max\{\text{dist}(x_0, \ell_P), \text{dist}(y_0, \ell_P)\} \lesssim \beta_{\mu,1}(P)\ell(P)$$

and

$$\max\{\text{dist}(x_0, \ell_R), \text{dist}(y_0, \ell_R)\} \lesssim \beta_{\mu,1}(R)\ell(R).$$

Then, show that both ℓ_P and ℓ_R are close to the line spanned by the segment $[x_0, y_0]$, hence close to each other. The details are an exercise. □

The lemma above is the only place, where the smoothness of the measure μ is required. Now, for the proof of Lemma 5.6:

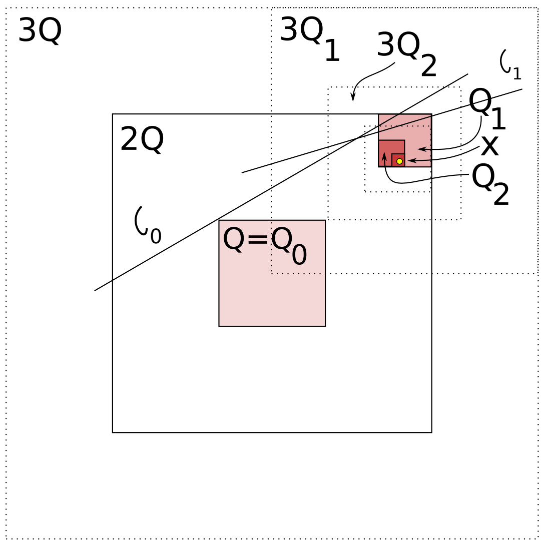

FIGURE 4. Possible cubes in the proof of Lemma 5.6

Proof of Lemma 5.6. Let $Q = Q_0, Q_1, Q_2, \ldots$ be a sequence of dyadic cubes such that $x \in Q_m \subset 2Q$ and $\ell(Q_m) = 2^{-m}\ell(Q)$ for all $m \geq 1$ (Note that the requirement $x \in Q_m$ may not be possible for $m = 0$, in case $x \in 2Q \setminus Q = 2Q_0 \setminus Q_0$.) One can now easily check, see Figure 4, that the cubes $P_m := 3Q_m$, $m \geq 0$, are nested, and

$$E \cap \frac{9}{10}P_m \cap \frac{9}{10}P_{m+1} \neq \emptyset, \quad m \geq 0. \tag{5.11}$$

For each $m \geq 0$, let ℓ_m be a line, which minimises $\beta_{\mu,1}(P_m, \ell)$ (so that $\ell_0 = \ell_Q$). Fix a constant $\theta > 0$ (whose value will only depend on the doubling of μ). Let $N \geq 0$ be the smallest number such that $\beta_{\mu,1}(P_N) \geq \theta$. If no such number exists, just let N be any (large) number. Let $a_N = c_{P_N} \in P_N$ be the μ-centre of mass in P_N (which evidently has positive measure by (5.11)). For $0 \leq m \leq N-1$, define a_m recursively to be the orthogonal projection of a_{m+1} to the line ℓ_m, so that $|a_{m+1} - a_m| = \text{dist}(a_{m+1}, \ell_m)$. Then

$$\text{dist}(x, \ell_Q) \leq |x - a_N| + \text{dist}(a_N, \ell_Q) \tag{5.12}$$

If $N = 0$, then $\beta_{\mu,1}(P_0) \geq \theta$, and the claim of the lemma is clear. So, in the sequel, assume that $N \geq 1$. Then, the term on the right hand side can be further estimated as follows:

$$\begin{aligned} \operatorname{dist}(a_N, \ell_Q) &\leq |a_N - a_{N-1}| + \operatorname{dist}(a_{N-1}, \ell_Q) \\ &= \operatorname{dist}(a_N, \ell_{N-1}) + \operatorname{dist}(a_{N-1}, \ell_Q) \leq \ldots \leq \sum_{m=1}^{N} \operatorname{dist}(a_m, \ell_{m-1}), \end{aligned} \tag{5.13}$$

recalling that $\ell_0 = \ell_Q$. The next task is to prove

$$\operatorname{dist}(a_m, \ell_{m-1}) \lesssim \beta_{\mu,1}(P_{m-1})\ell(P_{m-1}), \qquad 1 \leq m \leq N. \tag{5.14}$$

For $m = N$, this is simple. Since $a_N = c_{P_N}$, Lemma 5.7 with $p = 1$ and $\ell = \ell_{N-1}$ says that

$$\begin{aligned} |a_{N-1} - a_N| &= \operatorname{dist}(a_N, \ell_{N-1}) \\ &\overset{\text{L.5.7}}{\lesssim} \beta_{\mu,1}(P_N, \ell_{N-1})\ell(P_N) \\ &\lesssim \beta_{\mu,1}(P_{N-1}, \ell_{N-1})\ell(P_{N-1}) \\ &= \beta_{\mu,1}(P_{N-1})\ell(P_{N-1}) \leq \theta\ell(P_{N-1}). \end{aligned} \tag{5.15}$$

using also the doubling of μ (and (5.11)) in the second inequality, the definition of ℓ_{N-1} in the third, and the minimality assumption on N in the last. If $\theta > 0$ was chosen small enough, this implies that a_{N-1} is very close to a_N, and in particular

$$a_{N-1} \in \ell_{N-1} \cap P_{N-1}.$$

Next, *en route* to (5.14), the plan is to verify by backward induction that

$$a_m \in \ell_m \cap P_m, \qquad 0 \leq m \leq N-1, \tag{5.16}$$

which was just seen to be true for $m = N-1$. Suppose the claim is true for some $1 \leq m \leq N-1$. Then, by definition, a_{m-1} is the projection of $a_m \in \ell_m \cap P_m$ to the line ℓ_{m-1}, which minimises $\beta_{\mu,1}(P_{m-1})$. Applying Lemma 5.10 with

$$P = P_m, \quad R = P_{m-1}, \quad \text{and} \quad z = a_m \in \ell_m \cap P_m$$

(the lemma can be used because of (5.11)) gives

$$|a_m - a_{m-1}| = \operatorname{dist}(a_m, \ell_{m-1}) \lesssim \beta_{\mu,1}(P_{m-1})\ell(P_{m-1}) \leq \theta\ell(P_{m-1}). \tag{5.17}$$

Again, if $\theta > 0$ is small enough, this implies that a_{m-1} lies very close to a_m, and in particular inside P_{m-1}.

Now that (5.16) has been verified for all $0 \leq m \leq N-1$, the middle inequality in (5.17) is at our disposal for all $0 \leq m \leq N-1$ (this may feel a bit complicated, but we really need the information $a_m \in P_m$ to invoke Lemma 5.10, so we had to check that first). Combining this with (5.15) gives (5.14).

Combining (5.14) further with (5.13)-(5.15) yields

$$\operatorname{dist}(x, \ell_Q) \leq |x - a_N| + C \sum_{m=1}^{N} \beta_{\mu,1}(P_{m-1})\ell(P_{m-1}),$$

This is almost what we wanted. If $\beta(P_N) \geq \theta$, the term $|x - a_N|$ can be finally estimated by

$$|x - a_N| \lesssim \ell(P_N) \lesssim \frac{\beta(P_N)}{\theta}\ell(P_N),$$

Otherwise, if $\beta(P_m) < \theta$ for all $m \in \mathbb{N}$, one can just let $N \to \infty$, and the term $|x - a_N|$ vanishes. $\square$

Armed with the lemma, the proof of Theorem 5.5 is quite short:

Proof of Theorem 5.5. Fix $Q \in \mathcal{D}_E$ with $Q \subset R$, so that in particular $\operatorname{diam}(Q) \leq 10 \operatorname{diam}(E)$. Find a line ℓ_Q, which minimises $\beta^2_{\mu,1}(3Q, \ell)$, and then a point $x \in E \cap 2Q$, which maximises $\operatorname{dist}(x, \ell_Q)$. Then

$$\beta_{E,\infty}(2Q)\ell(Q) \leq \operatorname{dist}(x, \ell_Q) \lesssim \sum_{\substack{P \in \mathcal{D} \\ x \in P \subset 2Q}} \beta_{\mu,1}(3P)\ell(P)$$

by Lemma 5.6. Taking squares and using Cauchy-Schwarz leads to

$$\begin{aligned}\beta^2_{E,\infty}(2Q)\ell(Q)^2 &\leq \Big(\sum_{\substack{P \in \mathcal{D}_E \\ x \in P \subset 2Q}} \beta^2_{\mu,1}(3P)\ell(P)^{3/2} \Big) \Big(\sum_{\substack{P \in \mathcal{D}_E \\ x \in P \subset 2Q}} \ell(P)^{1/2} \Big) \\ &\lesssim \sum_{\substack{P \in \mathcal{D}_E \\ x \in P \subset 2Q}} \beta^2_{\mu,1}(3P)\ell(P)^{3/2}\ell(Q)^{1/2}.\end{aligned}$$

Then, dividing by $\ell(Q)$ and simply dropping the condition $x \in P$ gives

$$\beta^2_{E,\infty}(2Q)\ell(Q) \lesssim \sum_{\substack{P \in \mathcal{D}_E \\ x \in P \subset 2Q}} \beta^2_{\mu,1}(3P)\frac{\ell(P)^{3/2}}{\ell(Q)^{1/2}} \leq \sum_{\substack{P \in \mathcal{D}_E \\ P \subset 2Q}} \beta^2_{\mu,1}(3P)\frac{\ell(P)^{3/2}}{\ell(Q)^{1/2}}.$$

Next, sum over $Q \in \mathcal{D}_E$ with $Q \subset R$:

$$\begin{aligned}\sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset R}} \beta^2_{E,\infty}(2Q)\ell(Q) &\lesssim \sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset R}} \sum_{\substack{P \in \mathcal{D}_E \\ P \subset 2Q}} \beta^2_{\mu,1}(3P)\frac{\ell(P)^{3/2}}{\ell(Q)^{1/2}} \\ &\leq \sum_{\substack{P \in \mathcal{D}_E \\ P \subset 3R}} \beta^2_{\mu,1}(3P)\ell(P)^{3/2} \sum_{\substack{Q \in \mathcal{D} \\ 2Q \supset P}} \frac{1}{\ell(Q)^{1/2}} \\ &\lesssim \sum_{\substack{P \in \mathcal{D}_E \\ P \subset 3R}} \beta^2_{\mu,1}(3P)\ell(P).\end{aligned}$$

This proves Theorem 5.5. $\square$

5.2. **Proof of the L^1 traveling salesman theorem for smooth doubling measures.** The proofs of Theorems 5.2 and 5.3, for smooth measures, are now straightforward applications of Jones' L^∞ traveling salesman theorem, and the preceding machinery.

Proof of Theorem 5.2. Write $E := \operatorname{spt} \mu$, and let $R \in \mathcal{D}$ be the smallest dyadic cube such that $2R$ contains $3E$. Then $2R$ is the union of 9 dyadic squares $R_1, \ldots, R_9$, and for every j, it holds that $2R_j \subset 10E$. Then

$$\sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset 3E}} \beta^2_{E,\infty}(2Q)\ell(Q) = \sum_{j=1}^{9} \sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset R_j}} \beta^2_{E,\infty}(2Q)\ell(Q) \lesssim \sum_{\substack{P \in \mathcal{D}_E \\ P \subset 10E}} \beta^2_{\mu,1}(3P)\ell(P)$$

The "3" is easy to replace by "2". By the doubling hypothesis on μ, if $Q \in \mathcal{D}_E$ is the smallest cube such that $3P \subset 2Q$, then $\beta^2_{\mu,1}(3P) \lesssim \beta^2_{\mu,1}(2Q)$, and of course $2Q$ is still contained in, say, $20E$. So, the right hand side is finite by assumption, and now the existence of Γ follows from Jones' L^∞ traveling salesman Theorem 4.1. □

To prove Theorem 5.3, we recall an earlier exercise:

Exercise 5.18. Let $E \subset \mathbb{R}^2$ a uniformly 1-rectifiable compact set: for every ball B, the intersection $B \cap E$ can be covered by a continuum Γ_B of length $\mathcal{H}^1(\Gamma_B) \leq C\,\mathrm{diam}(B)$. Prove that there exists an AD regular continuum $\Gamma \supset E$ with $\mathrm{diam}(\Gamma) \sim \mathrm{diam}(E)$, where the implicit constant only depends on C.

Proof of Theorem 5.3. By the exercise, it suffices to prove that $E = \mathrm{spt}\,\mu$ is uniformly rectifiable: for every disc $B \subset \mathbb{R}^2$ there exists a continuum $\Gamma \supset B \cap E$ with $\mathcal{H}^1(\Gamma_B) \lesssim \mathrm{diam}(B)$. Here the implicit constants will only depend on the constant "C" in the assumed Carleson condition

$$\sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset R}} \beta^2_{\mu,1}(2Q)\ell(Q) \leq C\ell(R), \qquad R \in \mathcal{D}_E. \tag{5.19}$$

The plan is to apply Jones' traveling salesman theorem to the set $E_B := B \cap E$. Cover $3E_B$ by ~ 1 dyadic squares $R_j \in \mathcal{D}_E$ with $\ell(R_j) \leq \mathrm{diam}(B)$. It follows easily from Theorem 5.5, (5.19), and the inequality $\beta_{E_B,\infty}(2Q) \leq \beta_{E,\infty}(2Q)$, that

$$\sum_{\substack{Q \in \mathcal{D}_{E_B} \\ Q \subset 3E_B}} \beta^2_{E_B,\infty}(2Q)\ell(Q) \lesssim \sum_j \sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset 10R_j}} \beta^2_{\mu,1}(2Q)\ell(Q) \lesssim \mathrm{diam}(B).$$

Hence, by Jones' traveling salesman theorem, E_B can be covered by a continuum Γ_B with $\mathcal{H}^1(\Gamma_B) \lesssim \mathrm{diam}(B)$. □

5.3. **Proof of the L^1 traveling salesman theorem for general doubling measures.** In this section, I discuss the proof of Theorem 5.2, as given in the original paper [1], and without the "smoothness" assumption. The main difference to the previous proof is that there is no need for a reduction to Jones' L^∞ traveling salesman theorem: Badger and Schul use the $\beta_{\mu,1}$-numbers directly. The argument below (taking into account Section 7) may seem more complicated than the one above, but note that it works in all dimensions (and gives a better result). A caveat of the Badger-Schul approach seems to be that the 1-Ahlfors-David regularity of the curve is not so easy to prove (assuming the Carleson condition for the $\beta_{\mu,1}$-numbers).

The main component in the proof of Badger and Schul is the following "geometric traveling salesman theorem":

Theorem 5.20. *Let $n \geq 2$, $A > 1$, $x_0 \in \mathbb{R}^n$ and $r_0 > 0$. Let $(V_k)_{k\in\mathbb{N}}$ be a sequence of non-empty finite subsets of $B(x_0, Ar_0)$ such that the following conditions are satisfied:*

(V_{sep}) *The distance between distinct points in V_k is at least $2^{-k}r_0$.*
($V^\downarrow$) *For all $v \in V_k$, there exists $v^\downarrow \in V_{k+1}$ with $|v - v^\downarrow| < A2^{-(k+1)}r_0$.*
($V^\uparrow$) *For all $v \in V_{k+1}$, there exists $v^\uparrow \in V_k$ with $|v - v^\uparrow| < A2^{-k}r_0$.*

Further, assume that for all $k \geq 1$ and for all $v \in V_k$ there is a line $\ell_v = \ell_{k,v} \subset \mathbb{R}^n$ and a number $\alpha_v = \alpha_{k,v} \geq 0$ such that

$$\sup_{x \in (V_{k-1} \cup V_k) \cap B(v, 65A2^{-k}r_0)} \operatorname{dist}(x, \ell_v) \leq \alpha_v 2^{-k} r_0. \tag{5.21}$$

Then the sets V_k converge in the Hausdorff metric to a compact set $V \subset \overline{B(x_0, Ar_0)}$, and there exists a compact, connected set $\Gamma \subset \overline{B(x_0, Ar_0)}$ such that $\Gamma \supset V$, and

$$\mathcal{H}^1(\Gamma) \lesssim_{A,n} r_0 + \sum_{k \in \mathbb{N}} \sum_{v \in V_k} \alpha_v^2 2^{-k} r_0. \tag{5.22}$$

The proof of Theorem 5.20 is given in Section 7.

Proof of Theorem 5.2, assuming Theorem 5.20. Without loss of generality, we may assume that $\operatorname{spt} \mu$ is contained in a single dyadic cube $Q_0 \in \mathcal{D}$ with $\ell(Q_0) \lesssim \operatorname{diam}(\operatorname{spt} \mu)$. In any case, at most 2^n cubes with this property are needed, and we can construct 2^n separate curves inside each of those; in the end, to get a single curve, the resulting 2^n curves are simply joined with line segments of length $\lesssim \operatorname{diam}(\operatorname{spt} \mu)$.

Let $\mathcal{T} := \mathcal{D}_{\operatorname{spt} \mu}$ be the collection of dyadic cubes intersecting the support of μ, namely

$$\mathcal{T} := \{Q \subset Q_0 : Q \cap \operatorname{spt} \mu \neq \emptyset\}.$$

Then certainly $\mu(2Q) > 0$ for all $Q \in \mathcal{T}$, and we may define c_{2Q} as the centre of mass of μ in $2Q$:

$$c_{2Q} := \frac{1}{\mu(2Q)} \int_{2Q} x \, d\mu x \in 2Q$$

Write $r_0 := 2^{-k_0} := \ell(Q_0)$. For $k \geq 0$, let V_k be a maximal $2^{-(k+k_0)}$-separated set in

$$\{c_{2Q} : Q \in \mathcal{T} \cap \mathcal{D}_{k+k_0}\}.$$

It is then clear that V_k satisfies the separation condition (V_{sep}) of Theorem 5.20. The properties $(V^{\uparrow})$ and $(V_{\downarrow})$ are also fairly clear. To check $(V_{\downarrow})$, for instance, fix $v = c_{2Q} \in V_k$. Then Q clearly has a child $Q' \in \mathcal{T} \cap \mathcal{D}_{k+1}$, so either $v' = c_{2Q'} \in V_{k+1} \cap 2Q'$, or then there is some other point $v'' \in V_{k+1}$ at distance $\leq 2^{-(k+k_0)+1}$ from v'. In both cases, v is at distance $\lesssim 2^{-(k+k_0)}$ **some** point in V_{k+1}, which can then be designated as $v^{\downarrow}$. The proof of condition $(V^{\uparrow})$ is similar, and left for the reader.

Now we would like to define the lines ℓ_v and the numbers α_v for $v \in V_k$, $k \geq 1$. So, fix $v = c_{2Q} \in V_k$, $k \geq 1$. By the estimate (5.21), the line ℓ_v ought to be chosen so that $\operatorname{dist}(x, \ell_v)$ is nicely under control for **all** $x \in V_{k-1} \cap V_k$ nearby v. To do this, we consider a certain cube $\widehat{Q} \supset Q$, which is so large that $2\widehat{Q}$ contains not only $2Q$, but also $2Q'$ for **all** $Q' \in \mathcal{T} \cap [\mathcal{D}_k \cup \mathcal{D}_{k-1}]$ with

$$c_{2Q'} \in [V_{k-1} \cup V_k] \cap B(v, 65A2^{-(k+k_0)}).$$

Here $A \geq 1$ is any (dimensional) constant so that $(V^{\downarrow})$ and $(V^{\uparrow})$ hold. While $\widehat{Q}$ needs to be significantly larger than Q, it is clear that it can be chosen so that

$$\ell(\widehat{Q}) \leq A'\ell(Q) = A'2^{-(k+k_0)} \tag{5.23}$$

for some (dimensional) constant $A' \geq 1$. Now we can define ℓ_v. Let ℓ_v be any line ℓ such that

$$\beta_{\mu,p}(2\widehat{Q}, \ell) \leq 2\beta_{\mu,p}(2\widehat{Q}) =: \alpha_v. \tag{5.24}$$

Now, to estimate $\mathrm{dist}(x,\ell_v)$ for $x \in [V_{k-1} \cup V_k] \cap B(v, 65A2^{-(k+k_0)})$, we use the "centre of mass lemma", Lemma 5.7, which also played a role in the proof of Lemma 5.6. Fix $x = c_{2Q'} \in [V_{k-1} \cup V_k] \cap B(v, 65A2^{-(k+k_0)})$, so that $2Q' \subset 2\widehat{Q}$, and first observe that

$$\begin{aligned}\mathrm{dist}(x,\ell_v)^p &\overset{\text{L. 5.7}}{\leq} \beta_{\mu,p}(2Q',\ell_v)\,\mathrm{diam}(2Q')\\ &= \frac{1}{\mu(2Q')}\int_{2Q'} \mathrm{dist}(x,\ell_v)^p\,d\mu x\\ &\leq \frac{1}{\mu(2Q')}\int_{2\widehat{Q}} \mathrm{dist}(x,\ell_v)^p\,d\mu x.\end{aligned}$$

Next, since $Q' \in \mathcal{T}$ contains a point of $\mathrm{spt}\,\mu$, we have

$$\mu(2Q') \gtrsim_{D_\mu,n} \mu(2\widehat{Q}),$$

where the implicit constants depend on the doubling of μ, and A' from (5.23). It follows that

$$\mathrm{dist}(x,\ell_v)^p \lesssim_{D_\mu,n} \frac{1}{\mu(2\widehat{Q})}\int_{2\widehat{Q}} \mathrm{dist}(x,\ell_v)^p\,d\mu x = \beta_{\mu,p}(2\widehat{Q},\ell_v)^p\,\mathrm{diam}(2\widehat{Q})^p.$$

Recalling (5.24),

$$\mathrm{dist}(x,\ell_v) \lesssim \alpha_v \cdot \mathrm{diam}(2\widehat{Q}) \overset{(5.23)}{\lesssim_{D_\mu,n}} \alpha_v \cdot 2^{-(k+k_0)}.$$

Now, it remains to apply Theorem 5.20. Since $\mathrm{spt}\,\mu$ was assumed compact, it is evident that the sets V_k converge to $\mathrm{spt}\,\mu$ in the Hausdorff metric. So, according to Theorem 5.20, the support of μ can be covered by a single curve Γ of length

$$\mathcal{H}^1(\Gamma) \lesssim r_0 + \sum_{k\in\mathbb{N}}\sum_{v\in V_k} \alpha_v^2 2^{-(k+k_0)} \lesssim_{D_\mu,n} \mathrm{diam}(\mathrm{spt}\,\mu) + \sum_{k\in\mathbb{N}}\sum_{v\in V_k} \beta_{\mu,p}^2(2\widehat{Q}_v)2^{-(k+k_0)}.$$

Here $\widehat{Q}_v$ is, of course, the cube $\widehat{Q}$ associated with Q, if $v = c_{2Q}$. To complete the proof, it suffices to note that (1) all the cubes $\widehat{Q}_v$ arising this way are contained in $\lambda[\mathrm{spt}\,\mu]$ from some dimensional constant $\lambda \geq 1$, and (2) every cube $\widehat{Q}_v$ is only repeated a bounded number of times in the sum above (where "bounded" depends only on n). Hence,

$$\mathcal{H}^1(\Gamma) \lesssim_{D_\mu,n} \mathrm{diam}(\mathrm{spt}\,\mu) + \sum_{Q\subset\lambda[\mathrm{spt}\,\mu]} \beta_{\mu,p}^2(2Q)\ell(Q),$$

and the proof of Theorem 2.11 is complete, except for the geometric part in Theorem 5.20 (where the main work lies, of course). □

6. Rectifiability of sets and measures, and the Cauchy transform

In this section, we discuss various connections between rectifiability, measures, and the Cauchy transform.

6.1. The theorem of Mattila-Melnikov-Verdera.

The first goal is to prove the theorem of Mattila, Melnikov and Verdera [13] from 1996:

Theorem 6.1. *Let $E \subset \mathbb{C}$ be a 1-AD regular set such that the Cauchy transform associated to $\mathcal{H}^1|_E$ is bounded on $L^2(\mathcal{H}^1|_E)$. Then, the set E is uniformly 1-rectifiable.*

Other items on the menu are a theorem of G. David in Section 6.2, and the "Denjoy conjecture" (now a theorem) in Section 6.3.

To begin with, we briefly recall, what the "L^2-boundedness of the Cauchy transform" means. Let μ be a Radon measure on $\mathbb{C}$ For $\delta > 0$, let $\mathcal{C}_{\mu,\delta}$ be the operator formally defined by

$$\mathcal{C}_{\mu,\delta} f(z) := \int_{|z-w|\geq\delta} \frac{f(w)\, d\mu w}{z - w}.$$

In this section, we only need to define $\mathcal{C}_{\mu,\delta} f$ for bounded compactly supported functions $f \colon \mathbb{C} \to \mathbb{C}$, and then the integral above converges for every $z \in \mathbb{C}$. Now, the hypothesis that *the Cauchy transform associated to μ is bounded on $L^2(\mu)$* means, by definition, that

$$\|\mathcal{C}_{\mu,\delta} f\|_{L^2(\mu)} \leq C\|f\|_{L^2(\mu)}$$

for all bounded compactly supported functions $f \colon \mathbb{C} \to \mathbb{C}$, where C is a constant independent of $\delta > 0$; if μ is non-atomic, this actually **implies** that μ must have linear growth $\mu(B(x,r)) \lesssim r$, see Proposition 6.9 below, and then the definition of $\mathcal{C}_{\mu,\delta} f(z)$ makes sense for all $f \in L^2(\mu)$, $z \in \mathbb{C}$, by one application of the Cauchy-Schwarz inequality. This information will not be required in the current section, however.

A main idea behind the proof of Theorem 6.1 is the following striking equation, due to M. Melnikov: if μ is compactly supported and satisfies $\mu(B(x,r)) \lesssim r$, then

$$\|\mathcal{C}_{\mu,\delta}(1)\|_{L^2(\mu)}^2 = \tfrac{1}{6} c_\delta^2(\mu) + O(\mu(\mathbb{C})), \tag{6.2}$$

where $|O(\mu(\mathbb{C}))| \leq C\mu(\mathbb{C})$, and C only depends on the constants in the linear growth hypothesis $\mu(B(x,r)) \lesssim r$. The function 1 should be interpreted as $\chi_{\operatorname{spt}\mu}$, which is now bounded and compactly supported by hypothesis. I will not prove this inequality in the lecture notes, because it was discussed in the first half of the course by Henri; see Proposition 3.3 in Tolsa's book [17]. The quantity $c_\delta^2(\mu)$ is the (δ-truncated) Menger curvature

$$c_\delta^2(\mu) := \int_{|x-y|>\delta} \int_{|y-z|>\delta} \int_{|x-z|>\delta} c(x,y,z)^2 \, d\mu x \, d\mu y \, d\mu z,$$

where

$$c(x,y,z)^2 := \frac{\operatorname{dist}(z, L_{x,y})^2}{|z-x|^2|z-y|^2}, \qquad z \notin \{x,y\},$$

and $L_{x,y}$ is the line spanned by x and y for $x \neq y$. The numbers $c_\delta^2(\mu)$ above are positive and increase as $\delta \searrow 0$, so the limit $c^2(\mu) \in [1,\infty]$ exists (and has the obvious integral representation). Now, applying equation (6.2) to the restricted measures $\mu|_B$, where B is any bounded Borel set, yields

$$\|\mathcal{C}_{\mu,\delta}(\chi_B)\|_{L^2(\mu|_B)}^2 = \|\mathcal{C}_{\mu|_B,\delta}(1)\|_{L^2(\mu|_B)}^2 = \tfrac{1}{6} c_\delta^2(\mu|_B) + O(\mu(B)).$$

In particular, under the L^2-boundedness hypothesis of Theorem 6.1, we get

$$c_\delta^2(\mu|_B) \lesssim \|\mathcal{C}_{\mu,\delta}(\chi_B)\|_{L^2(\mu)}^2 + \mu(B) \lesssim \|\chi_B\|_{L^2(\mu)}^2 + \mu(B) \sim \mu(B).$$

Since this holds for all $\delta > 0$ uniformly, the conclusion is that

$$\iiint_{B\times B\times B} c(x,y,z)^2 \, d\mu x \, d\mu y \, d\mu z \lesssim \mu(B), \quad B \subset \mathbb{C}. \tag{6.3}$$

This is all the information we will need to prove the uniform rectifiability of μ (Theorem 6.1), so the precise form of the Cauchy transform actually plays a very small role in these lecture notes. Since 1-AD regular measures are evidently smooth and doubling, the plan is simply to deduce the Carleson condition

$$\sum_{\substack{Q \in \mathcal{D}_{\mathrm{spt}\,\mu} \\ Q \subset R}} \beta_{\mu,2}^2(2Q)\ell(Q) \lesssim \ell(R), \qquad R \in \mathcal{D}, \tag{6.4}$$

from (6.3), and then use Corollary 2.12 to infer that μ is uniformly rectifiable. In fact, the proof below will show that, for every smooth and doubling measure μ, the condition (6.3) implies

$$\sum_{\substack{Q \in \mathcal{D}_{\mathrm{spt}\,\mu} \\ Q \subset R}} \beta_{\mu,2}^2(2Q)\Theta(2Q)^3\ell(Q) \lesssim \mu(10R), \qquad R \in \mathcal{D}, \tag{6.5}$$

where $\Theta(2Q)$ is the density ratio $\Theta(Q) = \mu(2Q)/\ell(2Q)$. For 1-AD regular measures $\Theta(2Q) \sim 1$ for all $Q \in \mathcal{D}_{\mathrm{spt}\,\mu}$ and $\mu(10R) \lesssim \ell(R)$, so (6.5) immediately gives (6.4). I do not know, if (6.5) alone would imply uniform rectifiability for doubling measures.

To prove (6.5), write $\mathrm{spt}\,\mu =: E$. For $Q \in \mathcal{D}_E$, and any distinct points $x, y \in E \cap 10Q$, note that

$$\begin{aligned} \beta_{\mu,2}^2(2Q)\ell(2Q) &\leq \frac{\ell(2Q)}{\mu(2Q)} \int_{2Q} \frac{\mathrm{dist}(z, L_{x,y})^2}{\mathrm{diam}(2Q)^2}\, d\mu z \\ &\lesssim \frac{\ell(2Q)^3}{\mu(2Q)} \int_{2Q} \frac{\mathrm{dist}(z, L_{x,y})^2}{|z-x|^2|z-y|^2}\, d\mu z = \frac{\ell(2Q)^3}{\mu(2Q)} \int_{2Q} c(x,y,z)^2\, d\mu z. \end{aligned} \tag{6.6}$$

Since μ is smooth, there is a constant $c > 0$ such that the annulus

$$A_{x,Q} := B(x, \ell(Q)) \setminus B(x, c\ell(Q)) \subset 10Q \tag{6.7}$$

has measure $\mu(A_{x,Q}) \sim \mu(2Q)$ for all $x \in E \cap 2Q$. Consequently, averaging the inequality (6.6) over all $x \in E \cap 2Q$ and $y \in E \cap A_{x,Q} \subset E \cap 10Q$ gives

$$\beta_{\mu,2}^2(2Q)\ell(2Q) \lesssim \Theta(2Q)^{-3} \int_{x \in 2Q} \int_{y \in A_{x,Q}} \int_{z \in 2Q} c(x,y,z)^2\, d\mu x\, d\mu y\, d\mu z.$$

It remains to sum this inequality over all the cubes $Q \in \mathcal{D}_E$ with $Q \subset R$:

$$\begin{aligned} \sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset R}} \beta_{\mu,2}^2(2Q)\Theta(2Q)^3\ell(2Q) &\lesssim \sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset R}} \int_{x \in 2Q} \int_{y \in A_{x,Q}} \int_{z \in 2Q} c(x,y,z)^2\, d\mu x\, d\mu y\, d\mu z \\ &\leq \iint_{2R \times 2R} \Big[\sum_{\substack{Q \subset R \\ x \in E \cap 2Q}} \int_{y \in A_{x,Q}} c(x,y,z)^2\, d\mu y \Big]\, d\mu x\, x\mu z. \end{aligned}$$

Now, it suffices to note that for $x \in 2R$ fixed, the annuli $A_{x,Q}$ have bounded overlap as $Q \subset R$ varies in the collection of squares with $2Q \ni x$:

$$\sum_{\substack{Q \subset R \\ x \in E \cap 2Q}} \chi_{A_{x,Q}} \lesssim \chi_{10R}. \tag{6.8}$$

To see this, fix $y \in \mathbb{C}$. Assume that $y \in A_{x,Q}$ for some $Q \subset R$ with $x \in E \cap 2Q$. This forces $y \in 10Q \subset 10R$ by (6.7). Moreover, the requirement $y \in A_{x,Q}$ forces $\ell(Q) \sim |x-y|$, so there are only $\lesssim 1$ side-lengths $2^{-k} = \ell(Q)$ such that this can happen. And for any fixed side-length 2^{-k}, there are only $\lesssim 1$ squares Q with $\ell(Q) = 2^{-k}$ and $x \in 2Q$. Combining these facts, there are $\lesssim 1$ squares Q such that $\chi_{A_{x,Q}}(y) = 1$, and this gives (6.8).

All in all, we have now proven that

$$\sum_{\substack{Q \in \mathcal{D}_E \\ Q \subset R}} \beta_{\mu,2}^2(2Q)\Theta(2Q)^3 \ell(Q) \lesssim \iiint_{10R \times 10R \times 10R} c(x,y,z)^2 \, d\mu x \, d\mu y \, d\mu z \overset{(6.3)}{\lesssim} \mu(10R).$$

This is (6.5), so the proof of Theorem 6.1 is complete.

6.2. **David's theorem.** This short section contains the following result of G. David [5]:

Proposition 6.9. *Assume that μ is a non-atomic Radon measure, and $\mathcal{C}_\mu$ is bounded on $L^2(\mu)$. Then $\mu(B(x,r)) \leq C_2 r$ for all balls $B(x,r) \subset \mathbb{R}^2$, where C_2 only depend on the L^2-boundedness constant for $\mathcal{C}_\mu$.*

Remark 6.10. Note that the non-atomicity is essential. For instance, if $\mu = \delta_0$, then $\mathcal{C}_{\mu,\delta}$ is trivially bounded on $L^2(\mu)$ for all $\delta > 0$, because in fact $\mathcal{C}_{\mu,\delta}$ equals the zero-operator on $L^2(\mu)$ for any $\delta > 0$. Indeed,

$$\mathcal{C}_{\mu,\delta} f(0) = \int_{|w|>\delta} \frac{f(w)\, d\mu(w)}{w} = 0$$

regardless of $f \in L^2(\mu)$.

Proof of Proposition 6.9. Assume that $\|\mathcal{C}_{\mu,\delta}\|_{L^2(\mu)\to L^2(\mu)} \leq C_1$, $\delta > 0$, but μ **fails** to satisfy the uniform bound $\mu(B(x,r)) \leq 10C_2 r$, for some large C_2. This is used to show that μ has an atom, if C_2 is large enough, depending only on C_1.

Let $Q_0 \subset \mathbb{R}^2$ be some initial square, not necessarily dyadic, with

$$\Theta_0 := \Theta(Q_0) := \frac{\mu(Q_0)}{\ell(Q_0)} \geq C_2.$$

Now, the first step of the plan is to find a significantly smaller sub-square $Q_1 \subset Q_0$ with nearly the same mass as Q_0. More precisely, the claim is that for all $N \in \mathbb{N}$, there exists $Q_1 \subset Q_0$ with $\ell(Q_1) \leq \ell(Q_0)/2^N$, satisfying

$$\mu(Q_1) \geq \left(1 - \frac{A}{\Theta_0^2}\right)\mu(Q_0), \tag{6.11}$$

where $A = A(C_1, N) \geq 1$ is a suitable constant (the choice $N = 2$ will work for us in the end). To simplify notation slightly, assume that $Q_0 = [0,1]^2$, so $\ell(Q_0) = 1$. One may also assume that N is very large to begin with, because the claim is weaker for small N.

If (6.11) fails for all squares $Q_1 \subset Q_0$ with $\ell(Q_1) \leq \ell(Q_0)/2^N$, then in particular

$$\mu(2^{8N}Q_1 \cap Q_0) < \left(1 - \frac{A}{\Theta_0^2}\right)\mu(Q_0) \tag{6.12}$$

for all squares $Q_1 \subset Q_0$ with $\ell(Q_1) = 1/2^{10N}$ (since $\ell(2^{8N}Q_1) \leq 1/2^N$ for such squares Q_1, noting that in general $\ell(MQ) = (2M-1)\ell(Q)$ for $M > 1$). Now, pick $Q_1 \in \mathcal{D}_{10N} = \{Q \in \mathcal{D} : Q \subset Q_0 \text{ and } \ell(Q) = 2^{-10N}\}$ (using the pigeonhole principle) so that

$$\mu(Q_1) \geq \frac{\mu(Q_0)}{2^{20N}}. \tag{6.13}$$

Since (6.12) holds for Q_1,

$$\mu(Q_0 \setminus 2^{8N} Q_1) \geq \left(\frac{A}{\Theta_0^2}\right) \mu(Q_0),$$

which implies the existence of another square $Q_2 \in \mathcal{D}_{10N}$ with

$$Q_2 \subset Q_0 \setminus 2^{8N} Q_1 \quad \text{and} \quad \mu(Q_2) \geq \left(\frac{A}{\Theta_0^2}\right) \frac{\mu(Q_0)}{2^{20N}}. \tag{6.14}$$

In particular,

$$\operatorname{dist}(Q_1, Q_2) \gtrsim \ell(2^{8N} Q_1) \sim \frac{1}{2^N}. \tag{6.15}$$

At this point, consider $\mathcal{C}_{\mu,\delta}(\chi_{Q_2})(z)$ for $0 < \delta < \operatorname{dist}(Q_1, Q_2)$ and $z \in Q_1$:

$$\mathcal{C}_{\mu,\delta}(\chi_{Q_2})(z) = \int_{|w-z|\geq\delta} \frac{\chi_{Q_2}(w)}{z-w}\, d\mu w = \int_{Q_2} \frac{d\mu w}{z-w}.$$

For $z \in Q_1$ and $w \in Q_2$, the vectors $z - w$ have essentially constant direction, because the squares $Q_1, Q_2 \in \mathcal{D}_{10N}$ are tiny compared to their separation by (6.15), if N is large. In particular, N can be chosen so large (see computations below) that

$$|\mathcal{C}_{\mu,\delta}(\chi_{Q_2})(z)| \sim \int_{Q_2} \frac{d\mu w}{|z-w|} \gtrsim \mu(Q_2), \qquad z \in Q_1. \tag{6.16}$$

To prove (6.16) rigorously, let $z_0 \in Q_1$ and $w_0 \in Q_2$ be any points. Then,

$$\left| \int_{Q_2} \frac{d\mu w}{z-w} \right| \geq \left| \int_{Q_2} \frac{d\mu w}{z_0 - w_0} \right| - \int_{Q_2} \left| \frac{1}{z-w} - \frac{1}{z_0 - w_0} \right| d\mu w,$$

Since $z_0, w_0 \in Q_2 \subset Q_0 = [0,1]^2$, the first term on the right hand side is $\gtrsim \mu(Q_2)$, as desired in (6.16). For the second term, note that

$$\left| \frac{1}{z-w} - \frac{1}{z_0 - w_0} \right| \leq \frac{|z - z_0|}{|z-w||z_0 - w_0|} + \frac{|w - w_0|}{|z-w||z_0-w_0|} \lesssim 2^{2N} \cdot 2^{-10N} = \frac{1}{2^{8N}}$$

for $z \in Q_1$ and $w \in Q_2$, using (6.15). This gives (6.16) for large enough N.

It now follows from (6.16) that, and the L^2-boundedness hypothesis on $\mathcal{C}_\mu$, that

$$C_1 \mu(Q_2)^{1/2} \geq \|\mathcal{C}_{\mu,\delta}(\chi_{Q_2})\|_{L^2(\mu)} \gtrsim \mu(Q_2)\mu(Q_1)^{1/2}.$$

Combined with the measure estimates (6.13) and (6.14) (and recalling $\ell(Q_0) = 1$, which implies $\Theta_0 = \mu(Q_0)$), this gives

$$\frac{A^{1/2}}{2^{20N}} = \left(\frac{A}{\Theta_0^2}\right)^{1/2} \frac{\mu(Q_0)}{2^{20N}} \leq \mu(Q_1)^{1/2}\mu(Q_2)^{1/2} \lesssim C_1,$$

which is impossible for any choice of $A \gg C_1^2 2^{40N}$. The conclusion is that for some $A \sim C_1^2 2^{40N}$, there necessarily exists a square $Q_1 \subset Q_0$ with $\ell(Q_1) \leq \ell(Q_0)/2^N$, and satisfying (6.11).

To complete the proof of Proposition 6.9, the observation is iterated to find a sequence of (closed) squares $Q_0 \supset Q_1 \supset \ldots$, where the μ-measure decays so slowly that the intersection $\cap Q_j$ must be an atom for μ. More precisely, start with any (closed) square Q_0 with density $\Theta_0 \geq C_2$, as before, and assume that Q_j has been constructed for some j, with density $\Theta_j = \mu(Q_j)/\ell(Q_j) \geq C_2$. Apply the construction above with $N = 2$, say, to find $Q_{j+1} \subset Q_j$ with $\ell(Q_{j+1}) \leq \ell(Q_j)/2^2$ and

$$\Theta_{j+1} := \frac{\mu(Q_{j+1})}{\ell(Q_{j+1})} \geq \left(1 - \frac{A}{\Theta_j^2}\right) \frac{\mu(Q_j)}{\ell(Q_{j+1})} \geq 2^2 \left(1 - \frac{A}{\Theta_j^2}\right) \Theta_j.$$

Now, the key point is that **if** C_2 is sufficiently large (depending only on A, which only depends on C_1), then the inequality above shows that $\Theta_{j+1} \geq 2\Theta_j \geq C_2$. So, the construction can proceed, and one obtains $\Theta_{j+1} \geq 2\Theta_j$ for **all** $j \in \mathbb{N}$. Finally,

$$\mu(Q_j) \geq \left(1 - \frac{A}{\Theta_{j-1}^2}\right) \mu(Q_{j-1}) \geq \ldots \geq \left(1 - \frac{A}{\Theta_{j-1}^2}\right) \cdots \left(1 - \frac{A}{\Theta_0^2}\right) \mu(Q_0) \gtrsim \mu(Q_0)$$

for uniformly for all $j \geq 1$, because the infinite product

$$\prod_{j=0}^{\infty} \left(1 - \frac{A}{\Theta_j^2}\right)$$

converges to a positive number (because $\sum (A/\Theta_j^2) \lesssim_{C_1} \sum 100^{-j} < \infty$.) This proves that

$$\mu\left(\bigcap_{j \geq 0} Q_j\right) > 0,$$

as desired. □

6.3. **The Denjoy conjecture (aka Calderón's theorem).** In this section, which very closely follows (parts of) Section 4 in [17], we use the notation

$$\mathcal{C}_\delta \mu(z) := \mathcal{C}_{\mu,\delta} 1(z).$$

where μ is a finite measure. This allows us to view $\mathcal{C}_\delta$ as an operator acting on the space of complex measures $M(\mathbb{C})$.

The historical motivation for studying the connection between geometry, and the boundedness of the Cauchy transform, was to better understand *removable sets for bounded analytic functions*.

Definition 6.17. A compact set $E \subset \mathbb{C}$ is called *removable for bounded analytic functions*, or just *removable* if every bounded analytic function $f \colon \mathbb{C} \setminus E \to \mathbb{C}$ is constant.

In particular, every bounded analytic function $f \colon \mathbb{C} \setminus E \to \mathbb{C}$ can be extended to an entire function $f \colon \mathbb{C} \to \mathbb{C}$; this is why E is called "removable".

Subsets of lines with positive length are **not** removable; this is probably a "folklore" result, and I could not find a reference. As early as 1909, A. Denjoy attempted to prove the same for subsets of rectifiable curves (as opposed to subsets of lines). His proof contained a gap, and the statement became known as the Denjoy conjecture. It was resolved by A. Calderón in 1977, who proved that the Cauchy transform is bounded on Lipschitz graphs with sufficiently small constant. We have seen this result – even without the "small constant" restriction – on the course, in Henri's half. So, the main content of this section is to show, **why** Calderón's theorem implies the Denjoy conjecture.

Theorem 6.18 (Denjoy's conjecture, or Calderón's theorem)**.** *Let $\gamma \subset \mathbb{C}$ be a continuum of finite length, and let $E \subset \gamma$ be a compact subset of positive length. Then E is not removable.*

First of all, the property of being "non-removable" is clearly monotone: if $E_1 \subset E_2 \subset \mathbb{C}$ are compact, and E_1 is not removable, then E_2 is not removable either (just note that any non-constant bounded analytic function $f \colon \mathbb{C} \setminus E_1 \to \mathbb{C}$ is also non-constant on $\mathbb{C} \setminus E_2$, by basic properties of analytic functions).

Lemma 6.19. *Let γ be a continuum of finite length, and let $E \subset \gamma$ be a compact subset with $\mathcal{H}^1(E) > 0$. Then, there exists a Lipschitz graph Γ such that $\mathcal{H}^1(E \cap \Gamma) > 0$.*

Proof. Exercise. □

By the monotonicity of non-removability, Denjoy's conjecture follows, if we can prove the next theorem:

Theorem 6.20. *Let $\Gamma \subset \mathbb{C}$ be a Lipschitz graph, and let $E \subset \Gamma$ be a compact set with $\mathcal{H}^1(E) > 0$. Then E is not removable.*

By the results on the first half of the course (or see [17, Theorem 2.18]), we know the following:

Proposition 6.21. *Let $E \subset \Gamma$ is as in Theorem 6.20, and let $\mu := \mathcal{H}^1|_E$. Then the Cauchy transform maps $M(\mathbb{C})$ to $L^{1,\infty}(\mu)$ boundedly, which means, by definition, that*

$$\mu(\{x \in \mathbb{C} : |\mathcal{C}_\delta \nu(x)| > \lambda\}) \lesssim \frac{\|\nu\|}{\lambda}, \tag{6.22}$$

for all complex Borel measures ν, with implicit constants independent of $\delta > 0$.

The previous proposition is, by far, the hardest part in the proof of Theorem 6.20. Now, Theorem 6.20 will follow, if we just manage to prove the following proposition:

Proposition 6.23. *Assume that μ is a Radon measure with $E := \operatorname{spt} \mu$ compact, satisfying $\mu(B(x,r)) \lesssim r$ and the weak-$(1,1)$ bound (6.22). Then E is not removable. More precisely, there exists a function $h \colon E \to [0,1]$ with $\int h \, d\mu \gtrsim \mu(E)$, such that*

$$\mathcal{C}_\mu h(z) := \mathcal{C}(h \, d\mu)(z) := \int \frac{h(w) \, d\mu}{z - w}, \qquad z \in \mathbb{C} \setminus E,$$

defines a non-constant bounded analytic function on $\mathbb{C} \setminus E$.

The proof of the proposition requires two facts from functional analysis. The first is a corollary of the Hahn-Banach theorem:

Theorem 6.24. *Let $(V, \|\cdot\|)$ be a Banach space, and let $B_1, B_2 \subset V$ be disjoint non-empty convex subsets. Assuming that B_2 is open, there exists a number $r \in \mathbb{R}$ and a continuous linear map $\lambda \colon V \to \mathbb{C}$ such that*

$$\operatorname{Re} \lambda(x_1) > r \geq \operatorname{Re} \lambda(x_2) \quad \text{for all } x_1 \in B_1 \text{ and } x_2 \in B_2.$$

Proof. See Rudin's *Functional Analysis* [16], Theorem 3.4(a). □

The second fact identifies the space of complex Radon measures, namely $M(\mathbb{C})$, as the dual of a certain function space:

Theorem 6.25. *Let $C_0(\mathbb{C})$ be the vector space of all continuous function $\mathbb{C}$, which* vanish at infinity*: $\varphi \in C_0(\mathbb{C})$, if and only if φ is continuous, and for every $\epsilon > 0$, there exists $R_\epsilon > 0$ such that $|\varphi(x)| \leq \epsilon$ for $|x| \geq R_\epsilon$. Equipped with the usual* sup*-norm, $C_0(\mathbb{C})$ is a Banach space. The dual of $C_0(\mathbb{C})$ is the Banach space $M(\mathbb{C})$, equipped with the total variation norm. More precisely,* *if $\Lambda\colon C_0(\mathbb{C}) \to \mathbb{C}$ is a continuous linear map, then there exists measure $\nu = \nu_\Lambda \in M(\mathbb{C})$ such that*

$$\Lambda(\varphi) = \int \varphi \, d\nu, \qquad \varphi \in C_0(\mathbb{C}).$$

Proof. This is a version of the Riesz representation theorem, see Rudin's *Real and Complex Analysis*, Theorem 6.19. □

Finally, we define a slightly non-standard notion of *adjoint*. Assume that $T\colon M(\mathbb{C}) \to C_0(\mathbb{C})$ is a linear map. You should think that $T = \mathcal{C}_\delta$, although $\mathcal{C}_\delta$ need not quite map $M(\mathbb{C})$ to $C_0(\mathbb{C})$ (we will turn to this issue a bit later). Then, assume that there is another linear map $T^*\colon M(\mathbb{C}) \to C_0(\mathbb{C})$, satisfying the following relation:

$$\int (T\nu_1)\, d\nu_2 = \int (T^*\nu_2)\, d\nu_1 \tag{6.26}$$

for all $\nu_1, \nu_2 \in M(\mathbb{C})$. Then, we call T^* an *adjoint* of T. We are not claiming uniqueness, continuity, or any other properties usually associated with the notion of "adjoint". For $T = \mathcal{C}_\delta$, an adjoint is simply given by $T^* = -\mathcal{C}_\delta$, because

$$\begin{aligned}\int (\mathcal{C}_\delta \nu_1(z))\, d\nu_2 z &= \int \int_{|z-w|\geq\delta} \frac{d\nu_1 w}{z-w}\, d\nu_2 z \\ &= \int \left[-\int_{|z-w|\geq\delta} \frac{d\nu_2 z}{w-z} \right] d\nu_1 w = \int (-\mathcal{C}_\delta \nu_2(w))\, d\nu_1 w.\end{aligned}$$

Note that if μ is a measure such that the operators $T = \mathcal{C}_\delta$ satisfy (6.22), then clearly $T^* = -\mathcal{C}_\delta$ satisfies (6.22) with the same implicit constants.

With that in mind, we prove the following abstract variant of Proposition 6.23:

Proposition 6.27. *Assume that $T\colon M(\mathbb{C}) \to C_0(\mathbb{C})$ is a linear operator, and let $T^*\colon M(\mathbb{C}) \to C_0(\mathbb{C})$ be an adjoint satisfying (6.26). Assume that μ is a (positive) Radon measure with compact support $E := \operatorname{spt} \mu$ such that T^* maps $M(\mathbb{C})$ to $L^{1,\infty}(\mu)$ in the familiar sense that*

$$\mu(\{x \in \mathbb{C} : |T^*\nu(x)| > \lambda\}) \leq C\frac{\|\nu\|}{\lambda}, \qquad \nu \in M(\mathbb{C}). \tag{6.28}$$

Then, there exists a Borel function $h\colon E \to [0,1]$ such that $\|h\, d\mu\| \geq \|\mu\|/2$ and

$$\|T(h\, d\mu)\| < 3C.$$

Proof. Suppose that the claim fails: whenever $h\, d\mu$ lies in

$$G := \{f\, d\mu : f\colon E \to [0,1] \text{ is Borel, and } \|f\, d\mu\| \geq \|\mu\|/2\},$$

one has

$$T(h\, d\mu) \notin B_2 := \{g \in C_0(\mathbb{C}) : \|g\| < 3C\}.$$

Equivalently, $B_1 := T(G)$ is disjoint from B_2. Note that both B_1, B_2 are convex, and B_2 is open. By Theorem 6.24, there exists a continuous linear map $\Lambda\colon C_0(\mathbb{C}) \to \mathbb{C}$, which we may immediately identify with a measure $\nu = \nu_\Lambda \in M(\mathbb{C})$ by Theorem 6.25, such that

$$\operatorname{Re} \int \varphi_1\, d\nu > \operatorname{Re} \int \varphi_2\, d\nu$$

for all $\varphi_1 \in B_1$ and $\varphi_2 \in B_2$. Take an inf on the left hand side, and sup on the right hand side: recalling that $B_1 = T(F)$, this yields

$$\inf_{f\,d\mu \in G} \operatorname{Re} \int T(f\,d\mu)\,d\nu \geq \sup_{\varphi_2 \in B_2} \operatorname{Re} \int \varphi_2\,d\nu = 3C\|\nu\|.$$

Now, let $f := \chi_A$, where

$$A := \left\{ x \in E : |T^*\nu(x)| \leq \frac{2C\|\nu\|}{\|\mu\|} \right\}.$$

Then f is clearly a Borel function taking values in $[0,1]$. Moreover, by the main assumption (6.28),

$$\|f\,d\mu\| = \mu(A) = \|\mu\| - \mu(A^c) \geq \|\mu\| - \frac{C\|\nu\|}{2C\|\nu\|/\|\mu\|} = \frac{\|\mu\|}{2}$$

so $f\,d\mu \in F$. Hence

$$3C\|\nu\| \leq \operatorname{Re} \int T(f\,d\mu)\,d\nu \leq \int |T^*\nu| \cdot f\,d\mu \leq \frac{2C\|\nu\|}{\|\mu\|} \cdot \mu(A) \leq 2C\|\nu\|.$$

This is absurd, so the proof is complete. □

The only reason, why Proposition 6.27 does **not** imply Proposition 6.21 directly, is because $\mathcal{C}_\delta$ does not map $M(\mathbb{C})$ into $C_0(\mathbb{C})$. To fix this little technicality, we need to introduce a "smooth" version of $\mathcal{C}_\delta$. This is fairly standard.

6.3.1. *The smooth operators* $\tilde{\mathcal{C}}_\delta$. Let $\varphi \colon \mathbb{C} \to [0,\infty)$ be some smooth, non-negative radial function with $\int \varphi = 1$ and $\operatorname{spt} \varphi \subset B(0,1)$. Write $\varphi_\delta(z) := \delta^{-2}\varphi(x/\delta)$, so that $\int \varphi_\delta = 1$, and $\operatorname{spt} \varphi_\delta \subset B(0,\delta)$. Consider the kernel

$$\tilde{K}_\delta(z) := \frac{1}{z} * \varphi_\delta.$$

As the convolution of an L^1_{loc}-function, and a compactly supported function, the kernel $\tilde{K}_\delta$ is continous, and moreover

$$|\tilde{K}_\delta(z)| \leq \int \frac{|\varphi_\delta(w)|}{|z-w|}\,dw \leq \|\varphi_\delta\|_\infty \int_{\{|w|\leq\delta\}} \frac{dw}{|z-w|} \lesssim \frac{1}{\delta}, \qquad z \in \mathbb{C}.$$

Next, we claim that $\tilde{K}_\delta(z) = 1/z$ for $|z| \geq \delta$. To see this, write $\varphi_\delta(r)$ for the common value of φ_δ on the circle $S(0,r) := \{|w-0| = r\}$ (which exists by radiality). Since $w \mapsto 1/w$ is harmonic in $\mathbb{C} \setminus \{0\}$, the average of $1/w$ over any circle $S(z,r)$ **not** enclosing the origin equals $1/z$. This, combined with integration in polar coordinates gives

$$\begin{aligned}
\tilde{K}_\delta(z) &= \int \frac{\varphi_\delta(w)}{z-w}\,dw = c\int_0^\delta r \cdot \varphi_\delta(r) \left[\int_{S(0,r)} \frac{d\mathcal{H}^1(w)}{z-w} \right] dr \\
&= c\int_0^\delta r \cdot \varphi_\delta(r) \left[\int_{S(z,r)} \frac{d\mathcal{H}^1(w)}{w} \right] dr = \frac{c}{z} \int_0^\delta r \cdot \varphi_\delta(r) \cdot \mathcal{H}^1(S(0,r))\,dr \\
&= \frac{1}{z} \int \varphi_\delta(w)\,dw = \frac{1}{z}, \qquad \text{for } |z| \geq \delta.
\end{aligned}$$

Let $\tilde{\mathcal{C}}_\delta$ be the "singular" integral operator associated with $\tilde{K}_\delta$:

$$\tilde{\mathcal{C}}_\delta \nu(z) = \int \tilde{K}_\delta(z-w)\,d\nu w.$$

The fact that $\tilde{K}_\delta$ coincides with $1/z$ outside the ball $B(0,\delta)$ is very convenient for comparing the operators $\tilde{\mathcal{C}}_\delta$ and $\mathcal{C}_\delta$: if ν is a complex measure, then

$$\begin{aligned}|\tilde{\mathcal{C}}_\delta \nu(z) - \mathcal{C}_\delta \nu(z)| &= \left| \int \tilde{K}_\delta(z-w)\,d\nu w - \int_{|z-w|\geq\delta} \frac{d\nu w}{z-w} \right| \\ &\leq \int_{|z-w|\leq\delta} |\tilde{K}_\delta(z-w)|\,d|\nu|w \leq \|\tilde{K}_\delta\| \cdot |\nu|(B(z,\delta)) \\ &\lesssim \frac{|\nu|(B(z,\delta))}{\delta} \leq M(|\nu|)(z), \qquad z \in \mathbb{C}, \end{aligned} \tag{6.29}$$

where $M(|\nu|)$ is the "radial" maximal function $M(|\nu|)(a) := \sup_{\delta>0} \delta^{-1}|\nu|(B(z,\delta))$. Recall (or see [17, Theorem 2.5]) that the operator M maps $M(\mathbb{C})$ to $L^{1,\infty}(\mu)$ boundedly, whenever μ has linear growth (as in Proposition 6.23). So, if $\mathcal{C}_\delta$ also maps $M(\mathbb{C})$ to $L^{1,\infty}(\mu)$ boundedly – as assumed in Proposition 6.23 – the conclusion is that the smooth operator $\tilde{\mathcal{C}}_\delta$ does the same:

$$\mu(\{x \in \mathbb{C} : |\tilde{\mathcal{C}}_\delta \nu(x)| > \lambda\}) \lesssim \frac{\|\nu\|}{\lambda}. \tag{6.30}$$

The implicit constants are independent of $\delta > 0$. Note that an adjoint of $\tilde{\mathcal{C}}_\delta$ is again given by $(\tilde{\mathcal{C}}_\delta)^* = -\tilde{\mathcal{C}}_\delta$, repeating the computation from above Proposition 6.27. Now both $\tilde{\mathcal{C}}_\delta$ and $(\tilde{\mathcal{C}}_\delta)^*$ are linear operators mapping $M(\mathbb{C})$ to $C_0(\mathbb{C})$, and satisfying (6.30).

As a final lemma, we need a comparison between $\tilde{\mathcal{C}}_\delta$ and $\tilde{\mathcal{C}}_\epsilon$ for $0 < \epsilon \leq \delta$. The proof is so standard that we omit the details (see [17, Lemma 4.4]):

Lemma 6.31. *Let μ be a measure satisfying $\mu(B(x,r)) \lesssim r$, and let ν be any complex measure. Then, for $0 < \epsilon \leq \delta$,*

$$\|\tilde{\mathcal{C}}_\delta \nu\| \leq \|\tilde{\mathcal{C}}_\epsilon \nu\| + C\|M(|\nu|)\|,$$

where $C \geq 1$ is a constant depending only on the function φ.

We are finally in a position to prove Proposition 6.23.

Proof of Proposition 6.23. Let μ be a measure satisfying the hypotheses of the proposition, with $E = \operatorname{spt}\mu$ compact, so that (6.30) holds for $(\tilde{\mathcal{C}}_\delta)^* = -\tilde{\mathcal{C}}_\delta$ by the previous discussion. For $\delta > 0$, apply Proposition 6.27 to the operator $\tilde{\mathcal{C}}_\delta$: the result is a function $h_\delta \colon E \to [0,1]$ such that $\|h_\delta\,d\mu\| \geq \|\mu\|/2$, and $\|\tilde{\mathcal{C}}_\delta(h_\delta\,d\mu)\| \lesssim 1$, implicit constants independent of $\delta > 0$. By Lemma 6.31, we moreover have

$$\|\tilde{\mathcal{C}}_\delta(h_\epsilon\,d\mu)\| \leq \|\tilde{\mathcal{C}}_\epsilon(h_\epsilon\,d\mu)\| + C\|M(h_\epsilon\,d\mu)\| \lesssim 1 \tag{6.32}$$

uniformly for $0 < \epsilon \leq \delta$.

Since $L^\infty(\mu)$ is the dual of $L^1(\mu)$, the Banach-Alaoglu theorem states that the sequence $(h_\delta)_{\delta>0}$ has a weak*-convergent subsequence $(h_j)_{j\in\mathbb{N}} := (h_{\delta_j})_{j\in\mathbb{N}}$ with a limit $h \in L^\infty(\mu)$. This simply means that

$$\int h_j \cdot g\,d\mu \to \int h \cdot g\,d\mu, \qquad g \in L^1(\mu), \tag{6.33}$$

so in particular (applying the above with $g = 1$), one has $\|h\,d\mu\| \geq \|\mu\|/2$. Now, for $\delta > 0$ and $z \in \mathbb{C}$ fixed, apply (6.33) to $g(z) = \tilde{K}_\delta(z - w)$:

$$|\tilde{\mathcal{C}}_\delta(h\,d\mu)(z)| = \left| \int \tilde{K}_\delta(z-w)h(w)\,d\mu w \right|$$
$$= \lim_{j\to\infty} \left| \int \tilde{K}_\delta(z-w)h_j(w)\,d\mu w \right| \leq \limsup_{j\to\infty} \|\tilde{\mathcal{C}}_\delta(h_j\,d\mu)\| \lesssim 1.$$

The last estimate follows from (6.32). Finally, we infer from (6.29) that

$$\|\mathcal{C}_\delta(h\,d\mu)\| \lesssim \|\tilde{\mathcal{C}}_\delta(h\,d\mu)\| + \|M(h\,d\mu)\| \lesssim 1$$

uniformly in $\delta > 0$. Consequently, if $z \in \mathbb{C} \setminus E$, we have

$$|\mathcal{C}(h\,d\mu)(z)| = \left| \int_E \frac{h(w)\,d\mu w}{z-w} \right| = \lim_{\delta\to 0} \left| \int_{E\cap\{|z-w|>\delta\}} \frac{h(w)\,d\mu w}{z-w} \right| \lesssim 1.$$

which means that $z \mapsto \mathcal{C}(h\,d\mu)(z)$ defines a bounded analytic function on $\mathbb{C} \setminus E$. It is an exercise to check that the function is non-constant, hence E is non-removable. The proof of the proposition is complete. □

6.4. **Removability of the four corners Cantor set.** The purpose of the previous section was to prove that rectifiable sets of positive length are non-removable; now we will see that **some** purely unrectifiable sets of finite length – including the *four corners Cantor set*, depicted in Figure 5 – are removable. In fact, all purely unrectifiable sets of finite length

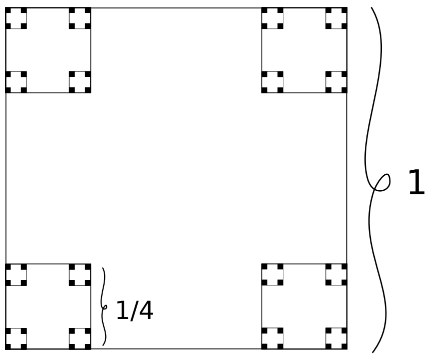

FIGURE 5. The four corners Cantor set.

are removable; this is a theorem of G. David [6] from 1998. The fact that **AD regular** purely unrectifiable sets are removable sets was already known a few years earlier. This follows by combining the Mattila-Melnikov-Verdera theorem (Theorem 1.2) with the following result [3, 4] of M. Christ from 1990:

Theorem 6.34 (Christ). *Let $E \subset \mathbb{C}$ be a compact AD regular set. If E is non-removable, then there exists an AD regular set $F \subset \mathbb{C}$ such that $\mathcal{H}^1(E \cap F) > 0$ so that the Cauchy transform associated with $\mu = \mathcal{H}^1|_F$ is bounded on $L^2(\mu)$. In particular (by the later theorem of Mattila-Melnikov-Verdera), $E \cap F$ is uniformly rectifiable.*

Since AD regular purely unrectifiable sets certainly cannot contain uniformly rectifiable pieces $E \cap F$ of positive length, they must be removable.

Unfortunately, the proof of David's, or even Christ's, theorem is too long for this course, so we need to take a hands-on approach, proving only the following rather special case.

Definition 6.35. Let $\alpha > 0$. A Radon measure ν on $\mathbb{C}$ is called *α-non-flat at z*, if $\operatorname{spt} \nu \cap B(z,r)$ is **not** contained in any cone centred at z, with opening angle α, for any $r > 0$. A measure is simply called *non-flat at z*, if it is α-non-flat at z for some $\alpha > 0$.

Theorem 6.36. *Let $E \subset \mathbb{C}$ be compact and AD regular, and write $\mu := \mathcal{H}^1|_E$. Assume that for μ almost every $a \in \mathbb{C}$, every tangent measure $\nu \in \operatorname{Tan}(\mu, a)$ is non-flat at 0. Then E is removable.*

This theorem, which is a slightly easier variant of Theorem 19.17 in Mattila's book [12] (our exposition follows his), quite clearly implies that the four corners Cantor set is removable. It would be easy to relax the hypotheses somewhat: the AD regularity could be replaced by positive lower 1-density $\mathcal{H}^1$ almost everywhere on E, and the "non-flatness" could be relaxed to "support not contained on a line".

The proof of Theorem 6.36 requires various preliminary results. The first states that a bounded analytic function $f \colon \mathbb{C} \setminus E \to \mathbb{C}$, which vanishes at infinity, is representable as the Cauchy transform of a complex measure:

Lemma 6.37. *Let $E \subset \mathbb{C}$ be a compact set with $\mathcal{H}^1(E) < \infty$, and let $f \colon \mathbb{C} \setminus E \to \mathbb{C}$ be analytic with $\|f\|_\infty \leq 1$ and*

$$f(\infty) := \lim_{z \to \infty} f(z) = 0.$$

Then, there exists a measure $\sigma \in M(\mathbb{C})$ with $\operatorname{spt} \sigma \subset E$ such that $|\sigma(B(x,r))| \leq r$ for all discs $B(x,r)$, and

$$f(z) = \mathcal{C}(\sigma)(z) = \int \frac{d\sigma w}{z - w}, \qquad z \in \mathbb{C} \setminus E.$$

Moreover, $\sigma = \varphi \cdot \mathcal{H}^1|_E$ for some function $\varphi \colon E \to \mathbb{C}$ with $\|\varphi\|_{L^\infty(\mathcal{H}^1)} \leq 1$.

Proof. Contained in Janne's presentation. □

Now, in the proof of Theorem 6.36, we start with a counter assumption: E is not removable. Then there is a non-constant bounded analytic function $f \colon \mathbb{C} \setminus E \to \infty$. The limit $f(\infty)$ exists by elementary theory of analytic functions (note that $z \mapsto f(1/z)$ is defined in a neighbourhood of the origin, and has a removable singularity at the origin), and $g = (f - f(\infty))/\|f\|_\infty$ is a a non-constant analytic function satisfying the assumptions of the previous lemma. Let

$$\sigma = \varphi \, d\mu = \varphi \cdot \mathcal{H}^1|_E$$

be the measure given by the lemma, associated with g; note that $\varphi \not\equiv 0$, because $g \not\equiv 0$. Then also $|\sigma| = |\varphi| \, d\mu$, from which is follows that

$$|\sigma|(B(x,r)) \lesssim r, \qquad x \in \mathbb{C}, \ r > 0. \tag{6.38}$$

Since μ is AD regular, every tangent measure $\nu \in \operatorname{Tan}(\mu, a)$, $a \in E$, has the form

$$\nu = c \cdot \lim_{i \to \infty} r_i^{-1} T_{a,r_i\sharp}\mu, \tag{6.39}$$

where $c > 0$, and $(r_i)_{i \in \mathbb{N}}$ is a sequence of positive radii tending to zero as $i \to \infty$. To see this, fix $a \in E$, and let $\nu = \lim_{i \to \infty} c_i T_{a,r_i\sharp}\mu \in \operatorname{Tan}(\mu, a)$. Then, because $\nu \neq 0$ by assumption (trivial measures are excluded from the definition of tangent measures), we find some some $R > 0$ such that

$$0 < \nu(U(0,R)) \leq \liminf_{i\to\infty} c_i\mu(U(a,Rr_i)) \lesssim \limsup_{i\to\infty}[c_iRr_i]$$
$$\lesssim \limsup_{i\to\infty} c_i\mu(B(a,Rr_i)) \leq \nu(B(0,R)) < \infty.$$

This implies that the the numbers c_ir_i lie, for large enough $i \in \mathbb{N}$, on some compact interval $[a,b] \subset (0,\infty)$, and hence there is a subsequence $c_{i_j}r_{i_j} \to c \in [a,b]$. Then, it is easy to check that $\nu = c \cdot \lim_{j\to\infty} r_{i_j}^{-1} T_{a,r_{i_j}\sharp}\mu$, as claimed in (6.39).

Now, if $\nu \in \mathrm{Tan}(\mu,a)$ as in (6.39), and φ (as in $\sigma = \varphi\, d\mu$) is non-vanishing and continuous in a neighbourhood of a, it is easy to see (exercise) that

$$\nu = \tilde{c} \cdot \lim_{i\to\infty} r_i^{-1} T_{a,r_i\sharp}\sigma \tag{6.40}$$

with $\tilde{c} = c/\varphi(a)$. In general, applying the Lebesgue differentiation theorem to φ, one can prove that at $|\sigma|$ almost every point $a \in \mathbb{C}$, every tangent measure $\nu \in \mathrm{Tan}(\mu,a)$ has the form (6.40); since σ is a non-trivial measure, "$|\sigma|$-almost every point" implies "μ-positively many points".

Further, it follows from (6.39) every $\nu \in \mathrm{Tan}(\mu,a)$, $a \in \mathrm{spt}\,\mu$, is AD regular: if $U(x,r)$ any open ball (centred on $\mathrm{spt}\,\mu$ or not), then

$$\nu(U(x,r)) \leq c \liminf_{i\to\infty} r_i^{-1}\mu(U(a+x,rr_i)) \lesssim c \cdot \liminf_{i\to\infty} \frac{rr_i}{r_i} = cr,$$

For the converse inequality, note that if $x \in \mathrm{spt}\,\nu$ and $r > 0$, then $\nu(U(x,r/2)) > 0$ for all $r > 0$, and it follows from the estimate above that $U(a+x,rr_i/2) \cap E \neq \emptyset$ for all large enough indices i. By the AD regularity of μ, this implies $\mu(B(a+x,rr_i)) \gtrsim rr_i$, and so

$$\nu(B(x,r)) \geq c \limsup_{i\to\infty} r_i^{-1}\mu(B(a+x,rr_i)) \gtrsim cr.$$

It will also be useful to note that $0 \in \mathrm{spt}\,\nu$ for all $\nu \in \mathrm{Tan}(\mu,a)$, $a \in E$. To this end, fix ν as above, and $r > 0$:

$$\nu(B(0,r)) \geq c \limsup_{i\to\infty} r_i^{-1}\mu(B(a,rr_i)) \gtrsim cr > 0.$$

In summary, if μ is AD regular, then for following hold for all $a \in \mathrm{spt}\,\mu$:

- Every $\nu \in \mathrm{Tan}(\mu,a)$ has the form $\nu = c\lim r_i^{-1}T_{a,r_i\sharp}\mu$, with $c > 0$ and $r_i \to 0$.
- Every $\nu \in \mathrm{Tan}(\mu,a)$ is AD regular. In particular $\mathrm{spt}\,\nu \neq \mathbb{C}$.
- $0 \in \mathrm{spt}\,\nu$ for all $\nu \in \mathrm{Tan}(\mu,a)$.

Now, we will show that tangent measures interact well with the Cauchy transform: the maximal Cauchy transform of any tangent of μ is bounded at 0:

Lemma 6.41. *If $a \in E$ is such that $\nu \in \mathrm{Tan}(\mu,a)$ is a tangent measure as in (6.40), then*

$$\sup_{0<r<R<\infty} \left| \int_{B(0,R)\setminus B(0,r)} \frac{d\nu w}{w} \right| < \infty.$$

Proof. The claim will follow, if we find a dense set of radii $0 < r < R < \infty$, and a constant $C \geq 1$, such that

$$\left| \int_{B(0,R)\setminus B(0,r)} \frac{d\nu w}{w} \right| \leq C,$$

After this, for arbitrary radii $0 < r < R < \infty$, we can find sequences (r_i) and (R_i) from the dense collection such that $0 < r < r_i < R_i < R < \infty$, and $r_i \searrow r$ and $R_i \nearrow R$. Then the sets $B(0,R_i) \setminus B(0,r_i)$ converge in a monotone way to $U(0,R) \setminus B(0,r)$, and consequently

$$\left| \int_{B(0,R)\setminus B(0,r)} \frac{d\nu w}{w} \right| \leq \left| \int_{\partial B(0,R)} \frac{d\nu w}{w} \right| + \lim_{i\to\infty} \left| \int_{B(0,R_i)\setminus B(0,r_i)} \frac{d\nu w}{w} \right| \leq \frac{\nu(B(0,R))}{R} + C.$$

Since ν is (upper) AD regular, the right hand side has a uniform bound.

Now, fix $a \in E$ as in the hypotheses, and let

$$\nu = c \cdot \lim_{j\to\infty} r_j^{-1} T_{a,r_j\sharp}\sigma \in \mathrm{Tan}(\mu, a),$$

where $c \in \mathbb{C} \setminus \{0\}$ (note that c has the form "$\tilde{c} = c/\varphi(a)$" from (6.40), so c can have an imaginary part, if $\varphi(a)$ does; this will not affect anything). Next, let $0 < r < R < \infty$ be radii such that $\nu(S(0,r)) = 0 = \nu(S(0,R))$; this holds but all but countably many pairs $0 < r < R < \infty$ and is used to infer that if

$$\int_{B(0,R)\setminus B(0,r)} \psi \, d\nu = \lim_{j\to\infty} \int_{B(0,R)\setminus B(0,r)} \psi \, d\nu_j, \tag{6.42}$$

whenever ψ is a continuous function on $B(0,R) \setminus B(0,r)$ and $\nu_j \to \nu$ weakly.[6]

After this observation, we just compute as follows:

$$\begin{aligned}
\left| \int_{B(0,R)\setminus B(0,r)} \frac{d\nu w}{w} \right| &\overset{(6.42)}{\sim_c} \lim_{j\to\infty} \left| \frac{1}{r_j} \int_{B(0,R)\setminus B(0,r)} \frac{d(T_{a,r_j\sharp}\sigma)w}{w} \right| \\
&= \lim_{j\to\infty} \left| \frac{1}{r_j} \int_{B(a,Rr_j)\setminus B(a,rr_j)} \frac{d\sigma w}{(a-w)/r_j} \right| \\
&= \lim_{j\to\infty} \left| \int_{B(a,Rr_j)\setminus B(a,rr_j)} \frac{d\sigma w}{a-w} \right| \leq 2 \cdot \mathcal{C}^*(\sigma)(a).
\end{aligned}$$

Here $\mathcal{C}^*(\sigma)$ is the maximal Cauchy transform

$$\mathcal{C}^*(\sigma)(a) = \sup_{\delta>0} |\mathcal{C}_\delta(\sigma)(a)|.$$

(The numbers $C_\delta(\sigma)(a)$ are well-defined, since σ is compactly supported; the tangent measures are typically not compactly supported, which explains the need for the double truncation in the formulation of the lemma.) The proof of the lemma is now complete, as soon as we verify that

$$\mathcal{C}^*(\sigma)(a) \lesssim \|\mathcal{C}(\sigma)\|_{L^\infty(\mathbb{C}\setminus E)} + M(|\sigma|)(a), \tag{6.43}$$

where $M(|\sigma|)$ is the "radial" maximal function $M(|\sigma|)(a) = \sup_{r>0} |\sigma|(B(a,r))/r$, which is now uniformly bounded by (6.38). Also, recall that $\|\mathcal{C}_\sigma\|_{L^\infty(\mathbb{C}\setminus E)} = \|g\|_{L^\infty(\mathbb{C}\setminus E)} < \infty$ by assumption.

[6] The general principle here is the following: if $\nu_i \to \nu$ weakly, and B is any Borel set with $\nu(\partial B) = 0$, then $\nu_i(B) \to \nu(B)$. This is an exercise.

To prove (6.43), fix $\delta > 0$. We first claim that there exists $b \in B(a,\delta/2) \setminus E$ such that

$$\int_{B(a,\delta)} \frac{d|\sigma|w}{|b-w|} \lesssim M(|\sigma|)(a). \tag{6.44}$$

Assuming this for a moment, we can estimate as follows:

$$\begin{aligned}|\mathcal{C}_{\delta}(\sigma)(a) - \mathcal{C}(\sigma)(b)| &= \left| \int_{\mathbb{C}\setminus B(a,\delta)} \frac{d\sigma w}{a-w} - \int \frac{d\sigma w}{b-w} \right| \\ &\leq \int_{\mathbb{C}\setminus B(a,\delta)} \frac{|a-b|}{|a-w||b-w|}\, d|\sigma|w + \int_{B(a,\delta)} \frac{d|\sigma|w}{|b-w|}\end{aligned}$$

The second term is bounded by $M(|\sigma|)(a)$ by (6.44), while in the first term the crucial points to note are the following: $|a-b| \leq \delta$, and $|a-w| \sim |b-w|$ for all $w \in \mathbb{C} \setminus B(a,\delta)$, because $b \in B(a,\delta/2)$. So, the first term is comparable to

$$\delta \cdot \int_{\mathbb{C}\setminus B(a,\delta)} \frac{d|\sigma|w}{|a-w|^2} \lesssim \delta \cdot \sum_{j\geq 0} \frac{1}{2^{2j}\delta^2} \cdot |\sigma|[B(a,2^{j+1}\delta)) \setminus B(a,2^j\delta)] \lesssim M(|\sigma|)(a).$$

Since $\delta > 0$ was arbitrary, and $|\mathcal{C}(\sigma)(b)| \leq \|\mathcal{C}(\sigma)\|_{L^\infty(\mathbb{C}\setminus E)}$, this completes the proof of (6.43), modulo finding the point $b \in B(a,\delta/2) \setminus E$ satisfying (6.44). This is done by a simple averaging trick:

$$\frac{1}{\delta^2} \int_{B(a,\delta/2)} \left[\int_{B(a,\delta)} \frac{d|\sigma|w}{|b-w|} \right] d\mathcal{L}^2(b) = \frac{1}{\delta^2} \int_{B(a,\delta)} \left[\int_{B(a,\delta/2)} \frac{d\mathcal{L}^2(b)}{|b-w|} \right] d|\sigma|w \lesssim \frac{|\sigma|(B(a,\delta))}{\delta},$$

noting (in the inner integral) that $B(a,\delta/2) \subset B(w,2\delta)$, whenever $w \in B(a,\delta)$. The proof of the lemma is complete. □

A very useful property of tangent measures is that "taking tangents twice" does not add much information:

Lemma 6.45. *Let μ be a Radon measure on $\mathbb{R}^n$. Then, at μ almost every $a \in \mathbb{R}^n$, **every** tangent measure $\nu \in \operatorname{Tan}(\mu,a)$ has the following property: $\operatorname{Tan}(\nu,x) \subset \operatorname{Tan}(\mu,a)$ for all $x \in \operatorname{spt}\nu$.*

Taking this result for granted (it is Theorem 14.16 in [12]), we are prepared to prove Theorem 6.36:

Proof of Theorem 6.36. The idea is to find a special tangent measure $\lambda \in \operatorname{Tan}(\mu,a)$ with $\operatorname{spt}\lambda$ contained in a half-plane, and then apply Lemma 6.41 to produce a contradiction (against the counter assumption that E is **not** removable).

Start by choosing any tangent measure $\nu \in \operatorname{Tan}(\mu,a)$, with $a \in E = \operatorname{spt}\mu$ such that the conclusion of Lemma 6.45 is valid, and the hypotheses of Lemma 6.41 are valid. Since μ is AD regular, ν is also AD regular by the discussion before Lemma 6.41. Typically $\operatorname{spt}\nu$ is not compact, but anyway the support of a 1-AD regular measure cannot be $\mathbb{C}$. So, there is a point $z \in \mathbb{C} \setminus \operatorname{spt}\nu$, which then of course satisfies $\rho := \operatorname{dist}(z,\operatorname{spt}\nu) > 0$. Note that

$$U(z,\rho) \cap \operatorname{spt}\nu = \emptyset \quad \text{and} \quad \partial B(z,\rho) \cap \operatorname{spt}\nu \neq \emptyset, \tag{6.46}$$

see Figure 6. Then, pick a point $y \in \partial B(z,\rho) \cap \operatorname{spt}\nu$. Since ν is AD regular, there exists a tangent measure

$$\lambda \in \operatorname{Tan}(\nu,y) \subset \operatorname{Tan}(\mu,a),$$

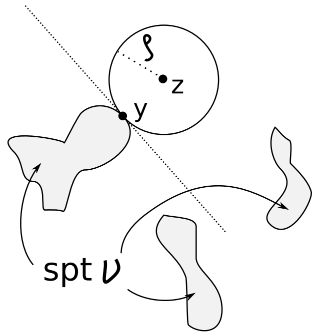

FIGURE 6. The choice of the points y and z.

using Lemma 6.45 in the second inclusion. In particular, $0 \in \operatorname{spt} \lambda$. Moreover, it follows from the first equation in (6.46) that the support of λ is entirely contained in some closed half-space H, with the $0 \in \partial H$ and $\partial H \perp (z - y)$. For simplicity of notation, assume that H is the lower half-plane $\{(x, y) : y \leq 0\} = \{w : \text{Im}\, w \leq 0\}$. Now, recall also the main assumption of the theorem: λ is α-non-flat at 0 for some $\alpha > 0$. In particular, this implies that $B(0, r) \operatorname{spt} \lambda$ is **not** contained in C_α for any $r > 0$, where C_α is the cone $C_\alpha := \{w : |\text{Im}\, w| \leq \alpha |w|\}$, see Figure 7. Since $\operatorname{spt} \lambda \subset H$, the conclusion is that for any $r > 0$, there exists a point $w \in B(0, r) \cap \operatorname{spt} \lambda$ with

$$w \in H \setminus C_\alpha = \{w : \text{Im}\, w < -\alpha |w|\} =: G_\alpha.$$

Note that if $w \in G_\alpha$, then $B(w, \alpha |w|/2) \subset G_{\alpha/2} = \{w : \text{Im}\, w < -\alpha |w|/2\}$.

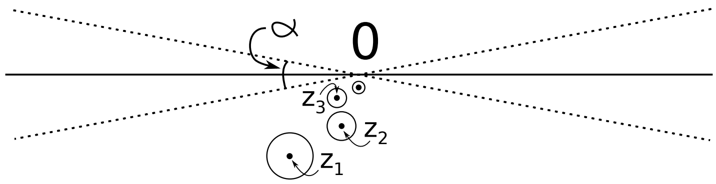

FIGURE 7. The half-space H and a few points $z_j \in \operatorname{spt} \lambda$.

With the observations above in mind, it is possible to find a sequence of points $\{w_j\}_{j \in \mathbb{N}}$ in $\operatorname{spt} \lambda$, converging to 0, such that the balls $B_j := B(w_j, \alpha |w_j|/2)$ are disjoint, and satisfy

$$B_j \subset G_{\alpha/2} \cap [\mathbb{C} \setminus B(0, |w_{k+1}|)], \qquad 1 \leq j \leq k. \tag{6.47}$$

Since $\lambda \in \operatorname{Tan}(\mu, a)$, and a was assumed to satisfy the hypotheses of Lemma 6.41, we have

$$\sup_{\epsilon > 0} \left| \int_{B(0,1) \setminus B(0,\epsilon)} \frac{d\lambda w}{w} \right| < \infty. \tag{6.48}$$

On the other hand, noting that $1/w = \bar{w}/|w|^2$, and using the fact that $\text{Im}\, w \leq 0$ for all $w \in \text{spt}\,\lambda$, plus the AD regularity of λ, we infer that

$$\left| \int_{B(1)\setminus B(0,|w_{k+1}|)} \frac{d\lambda w}{w} \right| \geq \int_{B(1)\setminus B(0,|w_{k+1}|)} \frac{\text{Im}\,\bar{w}}{|w|^2}\, d\lambda w \geq \sum_{j=1}^{k} \int_{B_j} \frac{-\text{Im}\, w}{|w|^2}\, d\lambda w$$

$$\overset{(6.47)}{\geq} \frac{\alpha}{2} \sum_{j=1}^{k} \int_{B_j} \frac{d\lambda w}{|w|} \gtrsim \alpha \sum_{j=1}^{k} \frac{\lambda(B_j)}{|w_j|} \sim \alpha k.$$

This contradicts (6.48) for large enough k, and the proof of the theorem is complete. □

7. The geometric construction of M. Badger and R. Schul

This section contains the proof of the geometric construction, Theorem 5.20, which is repeated below for convenience:

Theorem 7.1. *Let $n \geq 2$, $A > 1$, $x_0 \in \mathbb{R}^n$ and $r_0 > 0$. Let $(V_k)_{k\in\mathbb{N}}$ be a sequence of non-empty finite subsets of $B(x_0, Ar_0)$ such that the following conditions are satisfied:*

- (V_{sep}) *The distance between distinct points in V_k is at least $2^{-k}r_0$.*
- $(V^{\downarrow})$ *For all $v \in V_k$, there exists $v^{\downarrow} \in V_{k+1}$ with $|v - v^{\downarrow}| < A2^{-(k+1)}r_0$.*
- $(V^{\uparrow})$ *For all $v \in V_{k+1}$, there exists $v^{\uparrow} \in V_k$ with $|v - v^{\uparrow}| < A2^{-k}r_0$.*

Further, assume that for all $k \geq 1$ and for all $v \in V_k$ there is a line $\ell_v = \ell_{k,v} \subset \mathbb{R}^n$ and a number $\alpha_v = \alpha_{k,v} \geq 0$ such that

$$\sup_{x \in (V_{k-1}\cup V_k)\cap B(v,65A2^{-k}r_0)} \text{dist}(x, \ell_v) \leq \alpha_v 2^{-k} r_0. \tag{7.2}$$

Then the sets V_k converge in the Hausdorff metric to a compact set $V \subset \overline{B(x_0, Ar_0)}$, and there exists a compact, connected set $\Gamma \subset \overline{B(x_0, Ar_0)}$ such that $\Gamma \supset V$, and

$$\mathcal{H}^1(\Gamma) \lesssim_{A,n} r_0 + \sum_{k\in\mathbb{N}} \sum_{v\in V_k} \alpha_v^2 2^{-k} r_0. \tag{7.3}$$

To achieve a slight simplification in the proof, I record the following:

Proposition 7.4. *It suffices to prove Theorem 7.1 under the additional hypothesis that if either*

- *$k \in \mathbb{N}$ and $v, v' \in V_k$ and $w, w' \in V_k$ are distinct pairs of points, or*
- *$j < k$ and $v, v' \in V_j$ and $w, w' \in V_k$,*

then the intersection $[v, v'] \cap [w, w']$ has zero length.

Proof. It is clear that the extra hypothesis can be achieved by perturbing the points in the various sets V_k by arbitrarily small amounts. Note that these perturbations must be made so that no pair in V_j is also contained in V_k for any $k > j$. If every point of V_k is moved by less than $2^{-k}r_0$, then $(V_{sep}) - (V^{\uparrow})$ continue to hold. In case some of the numbers $\alpha_v = 0$, then any perturbation may cause (7.2) to fail, but the assumption $\alpha_v > 0$ can be made without loss of generality (check!). Finally, if the perturbations are small enough, the sets $\tilde{V}_k$ have the same limit set as the V_k's. Thus, it suffices to cover the limit set of the $\tilde{V}_k$'s by a continuum Γ satisfying (7.3). □

The convergence of the sequence V_k is, in fact, rather simple and based on (V_{III}) alone:

Proposition 7.5. *Let $V_0, V_1, \ldots$ be subsets of some fixed ball $B(x_0, Ar_0)$. If the sets V_k satisfy $(V^{\uparrow})$, then they converge to a compact set $V \subset \overline{B(x_0, Ar_0)}$ in the Hausdorff metric.*

Proof. Exercise (or read the paper of Badger and Schul). □

The statement of Theorem 7.1 is clearly "scaling invariant", i.e. one may assume

$$r_0 = 1.$$

Also, to avoid trivialities, I will assume that

$$\operatorname{card} V_k \geq 2, \qquad k \in \mathbb{N}.$$

Among other things, the next lemma defines a useful "ordering" for the points in $(V_k \cup V_{k-1}) \cap B(v, 65A2^{-k})$, assuming that the number $\alpha_{v,k}$ is sufficiently small. The lemma also gives some explanation for the "square" in α_v^2.

Lemma 7.6. *Let $0 \leq \alpha \leq 1/16$. Assume that $V \subset \mathbb{R}^n$ is a 1-separated set with $\operatorname{card} V \geq 2$, and there exist lines ℓ_1, ℓ_2 such that*

$$\operatorname{dist}(v, \ell_i) \leq \alpha, \qquad v \in V,\ i \in \{1, 2\}.$$

Let π_i be the orthogonal projection to ℓ_i. Then, one may identify both ℓ_1 ℓ_2 with $\mathbb{R}$ in such a way that

$$\pi_1(v) \leq \pi_1(v') \iff \pi_2(v) \leq \pi_2(v'), \qquad v, v' \in V.$$

Moreover, if v_1, v_2 are consecutive points relative to the order given by π_1 (equivalently π_2), then

$$\mathcal{H}^1([u_1, u_2]) < (1 + 3\alpha^2) \cdot \mathcal{H}^1([\pi_1(u_1), \pi_1(u_2)]), \qquad [u_1, u_2] \subset [v_1, v_2].$$

Also,

$$\mathcal{H}^1([u_1, u_2]) < (1 + 12\alpha^2) \cdot \mathcal{H}^1([\pi_1(u_1), \pi_1(u_2)]), \qquad [u_1, u_2] \subset \ell_2.$$

7.1. **Construction of the continuums.** In this section, the points of V_k will be covered by a (nearly) piecewise linear set Γ_k, whose connectedness and length will discussed in later sections. Each set Γ_k will consist of a finite number of *edges* $[v, v']$ between vertices $v, v' \in V_k$, plus a finite number of more complicated connected sets called *bridges*, to be defined presently.

For a vertex $v \in V_k$, define the *extension* $E(v) = E(v, k)$ inductively as follows. Let $v_0 = v \in E(v)$, and assume that v_j has been defined for some $j \geq 0$. Set $v_{j+1} := v_j^{\downarrow}$ (this was the closest "next generation" vertex to v_j). Then, define

$$E(v) := \overline{\bigcup_{j=0}^{\infty} [v_j, v_{j+1}]}.$$

Now, the *bridge* between two generation k vertices v, v' is

$$B(v, v') = B(v, v', k) := [v, v'] \cup E(v) \cup E(v').$$

Remark 7.7. In the special (but already interesting) case $V_k \subset V_{k+1} \subset \ldots$, the extension $E(v)$ simplifies to $\{v\}$ and $B(v, v') = [v, v']$.

7.1.1. *The induction begins.* In this tiny subsection, the initial curve Γ_0 is defined (which covers V_0). Consider a pair $v, v' \in V_0$. If $|v - v'| < 30A$, then $[v, v'] \subset \Gamma_0$. Otherwise, set $B(v, v') \subset \Gamma_0$. In other words,

$$\Gamma_0 := \bigcup_{|v-v'|<30A} [v, v'] \cup \bigcup_{|v-v'|\geq 30A} B(v, v').$$

7.1.2. *The construction of Γ_k based on Γ_{k-1}.* Here comes a key point: bridges stay, edges don't. Thus, if a bridge is contained in Γ_{k-1}, then it will also be contained in Γ_k. The edges of Γ_{k-1}, however, will be thrown away and replaced by new material (edges and bridges) in Γ_k. In symbols, Γ_k will look like this:

$$\Gamma_k := \bigcup_{v \in V_k} \Gamma_{k,v} \cup \bigcup_{j=0}^{k-1} \bigcup_{B(v',v'') \subset \Gamma_j} B(v', v''). \tag{7.8}$$

Here $\Gamma_{k,v}$ is a "local part" of Γ_k constructed inside the "neighbourhood"

$$\mathcal{N}(v) := B(v, 65A2^{-k}).$$

Note that $\mathcal{N}$ is the ball appearing in (7.2).

The local parts will look very different depending on whether α_v is "large" or "small". Here "small" simply means

$$0 < \alpha_v < \epsilon := 1/32,$$

and "large" means $\alpha_v \geq \epsilon$. The threshold $\epsilon = 1/32$ has been chosen so that Lemma 7.6 can be applied to any number smaller than 2ϵ.

Case (L)**.** Assume that $v \in V_k$ and $\alpha_v \geq \epsilon$. Let $v', v'' \in V_k \cap \mathcal{N}(v)$. If $|v' - v''| < 30A2^{-k}$, add the edge $[v', v'']$ to $\Gamma_{v,k}$. In the opposite case $|v' - v''| \geq 30A2^{-k}$, add the bridge $B(v', v'')$ to $\Gamma_{k,v}$. The case (L) is complete.

Here (L) obviously stands for "large". The "small" case (S) is more complicated and divides further into various sub-cases. However, before starting, a crucial point is worth emphasising:

Principle. At any stage of the construction, two points $v, v' \in V_k$ will be joined by and a finite sequence of edges in Γ_k **if and only if** $|v - v'| < 30A2^{-k}$. Check that this is the true for the cases above, and keep this in mind in the future!

Case (S)**.** Assume that $\alpha_v < \epsilon$ and note that the "ordering" Lemma 7.6 now applies to all points in $(V_{k-1} \cup V_k) \cap \mathcal{N}(v)$ and the line ℓ_v (once the picture is scaled by 2^k). In particular, once an orientation for ℓ_v has been fixed, it makes sense to write things like "v' is to the left from v''". I will also write $v' < v''$, if v' is to the left from v''.

With this ordering in mind, enumerate the points in $V_k \cap \mathcal{N}(v)$ from left to right as $v_{-l} < \ldots < v_{-1} < v_0 < v_1 < \ldots < v_m$, where $v_0 = v$. It may happen that v_0 is the only element on this list! I will begin by describing how the "right half" Γ_v^R of Γ_v looks like. This is the part of Γ_v, whose construction involves the points v_i with $i > 0$ (should any exist, which is neither clear nor assumed at this point). The "left half" will eventually be treated symmetrically.

Start from v and start moving right along the sequence $v_1, v_2, \ldots$ Include the edge $[v_i, v_{i+1}]$ to Γ_v^R as long as

$$|v_{i+1} - v_i| < 30A2^{-k} \quad \text{and} \quad v_{i+1} \in B(v, 30A2^{-k}). \tag{7.9}$$

(These conditions could easily hold for all the points $v_1, \ldots, v_m$). If one of the conditions eventually fails, **stop right there**! The construction will now divide into sub-cases depending on what happened.

Subcase (S-NT)**.** Here "NT" stands for "non-terminal", because this is the sub-case, where the algorithm above produced at least one edge. In other words $|v_1 - v| < 30A2^{-k}$, and the edge $[v, v_1]$ (and possibly much more) was added to Γ_v^R. The construction of Γ_v^R is complete in this simple sub-case.

Subcase (S-T)**.** Here "T" stands for "terminal", because this is the sub-case, where the algorithm above left us empty-handed: either v_1 does not exist at all, or $|v_1 - v| \geq 30A2^{-k}$, and no edges were added. In this case the vertex v will be called *terminal to the right* (*terminal to the left* will be defined similarly while constructing Γ_v^L). Now, the construction of Γ_v^R will depend on how the previous generation points $V_{k-1} \cap \mathcal{N}(v)$ are positioned. Again, since α_v is very small, and the points in $V_{k-1} \cap \mathcal{N}(v)$ lie at distance $\alpha_v 2^{-k}$ from ℓ_v, they can be arranged from "left to right" as

$$w_{-r} < \ldots w_{-1} < w_0 < w_1 < \ldots < w_s,$$

where $w_0 = v^\uparrow \in B(v, A2^{-k})$ is the closest point of V_{k-1} to v. It goes without saying that "left to right" means the same order as with the points v_i above. Let w_r be the right-most vertex on the list above, which still lies in $B(v, 2A2^{-k})$.[7] Consider the following two sub-sub-cases:

(S-TT) This stands for "terminal terminal", because the definition of Γ_v^R will simply be $\{v\}$. And what is this case? It occurs, if either $r = s$ (so everything in $V_{k-1} \cap \mathcal{N}(v)$ to the right from w_0 is contained in $B(v, 2A2^{-k})$), or then $|w_r - w_{r+1}| \geq 30A2^{-(k-1)}$, so there is no Γ_{k-1}-edge joining w_r and w_{r+1} (recall the **Principle**!).

(S-TB) This stands for "terminal bridge", because now – and only now – a bridge will be added. Since (S-TT) does not occur, the point w_{r+1} exists and satisfies

$$|w_r - w_{r+1}| \leq 30A2^{-(k-1)} = 60A2^{-k}.$$

By the assumption $(V^\downarrow)$, one moreover has $|w_{r+1} - w_{r+1}^\downarrow| < A2^{-k}$, which implies that

$$|w_{r+1}^\downarrow - v| \leq |v - w_r| + |w_r - w_{r+1}| + |w_{r+1} - w_{r+1}^\downarrow| < 2A2^{-k} + 60A2^{-k} + A2^{-k} = 63A2^{-k}.$$

The upshot is that $w_{r+1}^\downarrow \in V_k \cap \mathcal{N}(v) \setminus \{v\}$ ($w_{r+1}^\downarrow \neq v$, because it's quite far away; check!), and hence v_1 exists. But since we ended up in the "T-cases", we know that $|v - v_1| \geq 30A2^{-k}$. Now, a bridge $B(v, v_1)$ is added to Γ_v^R, and the construction of Γ_v^R is complete.

The construction of the "left part" Γ_v^L is symmetric. I make one last remark about Case (S-TB). Let v_1 and w_{r+1} be as above. It is useful to observe that while nothing necessitates that $v_1 = w_{r+1}^\downarrow$, we still have

$$|v_1 - w_{r+1}| < 2A2^{-k}. \tag{7.10}$$

Indeed, note that $v_1^\uparrow \in V_{k-1} \cap \mathcal{N}(v)$, hence $v_1^\uparrow$ must lie "to the right" from w_{r+1}, since w_{r+1} is the first vertex "to the right" from w_r. Now, if (7.10) failed, the situation would be as in Figure 8 below.

[7]This is no typo: the "2" should not be "30".

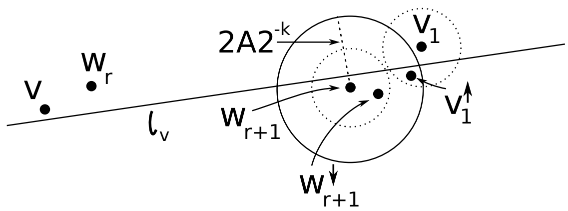

FIGURE 8. The position of the points $v_1, v_1^{\uparrow}, w_{r+1}, w_{r+1}^{\downarrow}$ in the absurd situation that (7.10) failed. Note how $w_{r+1}^{\downarrow}$ must lie to the left of v_1, which contradicts the definition of v_1 as the **first** vertex in V_k to the right from v.

But since $w_{r+1}^{\downarrow} \in V_k \cap \mathcal{N}(v)$, it should lie "to the right" from v_1, which is now evidently impossible, using $|w_{r+1}^{\downarrow} - w_r| < A2^{-k}$, the counterassumption $|v_1 - w_{r+1}| \geq 2A2^{-k}$, and the fact that $\alpha_v < \epsilon$ (think of how the proof would go precisely, if all the points actually lay on ℓ_v!). This proves (7.10).

The construction of Γ_v is complete, and the set Γ_k is obtained by performing the same algorithm for every point $v \in V_k$ (and keeping the old bridges, as indicated in (7.8)).

8. THE CONNECTEDNESS OF Γ_k

So far, all we (are supposed to) know is that $V_k \subset \Gamma_k$, and each pair $v, v' \in V_k$ with $|v - v'| \leq 30A2^{-k}$ is connected by a sequence of edges in Γ_k. In this section, we will verify that Γ_k is, indeed, a connected set containing V_k. Equivalently, an arbitrary pair of points $v, v' \in V_k$ can be connected by a *tour* inside Γ_k.[8]

The proof runs by induction on k. First, it is clear that Γ_0 is connected, and indeed every pair $v, v' \in V_0$ is connected in Γ_0 by either an edge or a bridge. Now, suppose (inductively) that every pair of points in V_{k-1} can be connected by a tour inside Γ_{k-1}. Then, fix $v_1, v_p \in \Gamma_k$ (the index parameter $p \in \mathbb{N}$ will be explained soon), and let $w_1 := v_1^{\uparrow}$ and $w_p := v_p^{\uparrow}$. By the inductive hypothesis, the points w_1 and w_p can be joined by a tour in Γ_{k-1}, namely

$$w_1, w_2, \ldots, w_p$$

where w_i is connected to w_{i+1} by either an edge or a bridge contained in Γ_{k-1}. Note that

$$|v_1 - w_1| < A2^{-k} \quad \text{and} \quad |v_p - w_p| < A2^{-k}$$

by condition $(V^{\uparrow})$. Now, suppose inductively that $1 \leq t \leq p-1$, and there exists a vertex $v_t \in V_k$ such that $|v_t - w_t| < A2^{-k}$, and v_1 is connected to v_t by a tour in Γ_k. This is clearly true for $t = 1$, and the whole proof is about showing the same for $t = p$. If $t = p - 1$, then $v_{t+1} = v_p$, and it satisfies $|v_{t+1} - w_{t+1}| < A2^{-k}$; if $t \leq p - 2$, simply let $v_{t+1} \in V_k$ be any point satisfying $|v_{t+1} - w_{t+1}| < A2^{-(k-1)}$ (which exists by $(V^{\downarrow})$). In both cases, it remains to demonstrate that v_t and v_{t+1} can be connected by a tour contained in Γ_k.

The proof divides into two cases, depending on whether w_t is connected to w_{t+1} by a bridge or an edge contained in Γ_{k-1}.

[8] By definition, a *tour* is a finite sequence of vertices $x_1, \ldots, x_p \in V_k$, where x_i is connected to x_{i+1} by either an edge or a bridge in Γ_k. Note that the bridge need not necessarily be of the form $B(x_i, x_{i+1})$: the requirement is simply that $B(x, y) \subset \Gamma_k$ and $x_i, x_{i+1} \in B(x, y)$.

Case (Bridge)**.** Assume that w_t is connected to w_{t+1} by a bridge contained in Γ_{k-1}. This means that there are vertices $x, y \in V_j$ for some $0 \leq j \leq k-1$ such that $w_t, w_{t+1} \in B(x,y)$. Let $v_t' := w_t^{\downarrow} \in B(x,y)$ and $v_{t+1}' := w_{t+1}^{\downarrow} \in B(x,y)$. It is not automatically clear that $v_t' = v_t$ and v_{t+1}', but it turns out that $|v_t - v_t'|$ and $|v_{t+1} - v_{t+1}'|$ are so small that there are connecting edges. For example,

$$|v_t - v_t'| \leq |v_t - w_t| + |w_t + v_t'| \leq 2A2^{-(k-1)} < 30A2^{-k},$$

which by the **Principle** implies that v_t is connected to v_t' by an edge in Γ_k. The same is true for the pair v_{t+1}, v_{t+1}'. Since v_t' and v_{t+1}' are connected by the bridge $B(x,y) \subset \Gamma_k$, it follows that v_t and v_{t+1} are connected by a tour in Γ_k.

Case (Edge)**.** Assume that w_t is connected to w_{t+1} by an edge in Γ_{k-1}. In particular, $|w_t - w_{t+1}| < 30C2^{-(k-1)} = 60C2^{-k}$ by the **Principle**, and consequently

$$|v_t - v_{t+1}| \leq |v_t - w_t + |w_t - w_{t+1}| + |w_{t+1} - v_{t+1}| < 2C2^{-k} + 60C2^{-k} + 2C^{-k} = 64C2^{-k}.$$

This means that $v_{t+1} \in \Gamma_k \cap \mathcal{N}(v_t)$ and vice versa. Consequently, if $\alpha_{v_t} \geq \epsilon$, then either $[v_t, v_{t+1}] \subset \Gamma_{v_t} \subset \Gamma_k$ or $B(v_t, v_{t+1}) \subset \Gamma_{v_t} \subset \Gamma_k$ by Case (L), and we are done.

Now, assume that $\alpha_{v_t} < \epsilon$, so $\Gamma_{v_t} \subset \Gamma_k$ is constructed by one of sub-cases in Case (S). As before, the assumption $\alpha_{v_t} < \epsilon$ means that the points in $V_k \cap \mathcal{N}(v_t)$ can be ordered "along" the line ℓ_{v_t}, and the notions of "left" and "right" make sense. Assume that v_{t+1} is "to the right" of v_t, and the points of $V_k \cap \mathcal{N}(v_t)$ lying between v_t and v_{t+1} are indexed as

$$v_t = z_1 < z_2 < \ldots < z_q = v_{t+1}.$$

Using the fact that $|v_t - v_{t+1}| < 64C2^{-k}$, and that α_{v_t} is very small, it is easy to convince oneself that

$$v_t, v_{t+1} \in B(z_i, 65A2^{-k}) = \mathcal{N}(z_i), \quad 1 \leq i \leq q.$$

(I omit the proof, because it is so clear that this holds for **some** suitable choice of ϵ, even if it this were not exactly $\epsilon = 1/32$). Consequently, if it happens that $\alpha_{z_i} \geq \epsilon$ for **even one** index $1 \leq i \leq q$, then either $[v_t, v_{t+1}] \subset \Gamma_{z_i} \subset \Gamma_k$ or $B(v_t, v_{t+1}) \subset \Gamma_{z_i} \subset \Gamma_k$ by Case (L), and we are happy.

So, the remaining case is where $\alpha_{z_i} < \epsilon$ for all $1 \leq i \leq q$. The plan is to show that z_i is connected to z_{i+1}, for $1 \leq i \leq q-1$, by either an edge or a bridge contained in $\Gamma_{z_i} \subset \Gamma_k$; it will then follow that v_t is connected to v_{t+1} by a tour in Γ_k, as claimed. If $|z_i - z_{i+1}| < 30A2^{-k}$, then we are in Case (S-NT) and $[z_i, z_{i+1}] \subset \Gamma_{z_i} \subset \Gamma_k$.

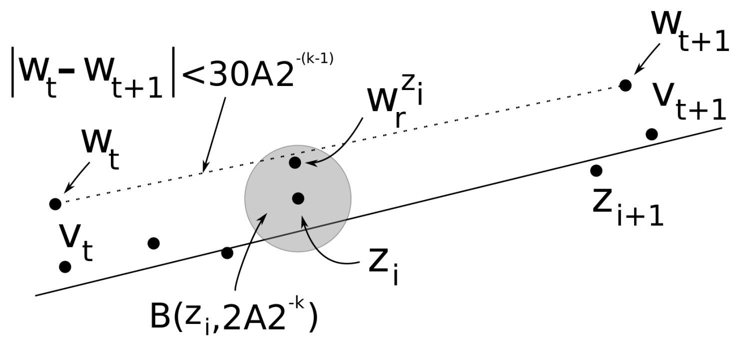

FIGURE 9. The final case in the proof of connectedness.

Otherwise, $|z_i - z_{i+1}| \geq 30A2^{-k}$, and we are in either Case (S-TT) or (S-TB). In Case (S-TB), the bridge $B(z_i, z_{i+1})$ is contained in Γ_k, and we are happy. So, it suffices to **rule out** Case (S-TT), where potentially $\Gamma_{z_i}^R = \{z_i\}$. This uses the fact that $[w_t, w_{t+1}]$ is an edge in Γ_{k-1}, so in particular $|w_t - w_{t+1}| < 30A2^{-(k-1)}$. Recall the point $w_r^{z_i} \in B(z_i, 2A2^{-k})$ from the Case (S-TT) associated with z_i: now this case could only occur, if the "next point of V_{k-1} to the right" from $w_r^{z_i}$" was very far away (at distance $\geq 30A2^{-(k-1)}$) or did not exist at all. But since z_i satisfies $v_t < z_i < v_{t+1}$, and

$$|z_i - v_{t+1}| \geq |z_i - z_{i+1}| \geq 30A2^{-k},$$

we can infer that $w_{t+1} \in V_{k-1} \cap \mathcal{N}(z_i)$ lies (strictly) to the right from $w_r^{z_i}$, and satisfies $|w_r^{z_i} - w_{t+1}| \leq |w_t - w_{t+1}| < 30A2^{-(k-1)}$. Hence also the "next point of V_{k-1} to the right from $w_r^{z_i}$" is at distance $< 30A2^{-(k-1)}$, and Case (S-TT) cannot occur at z_i. The proof of connectedness is complete.

9. Length estimates

Now we really arrive at the core of the proof. The aim will be to prove that

$$\mathcal{H}^1(\Gamma_k) \lesssim 1 + \sum_{j \leq k} \sum_{v \in V_j} \alpha_v^2 2^{-j}. \tag{9.1}$$

(Had we not normalised $r_0 = 1$, then r_0 would appear above in place of 1). The obvious first attempt would be to estimate

$$\mathcal{H}^1(\Gamma_k) \leq \mathcal{H}^1(\Gamma_{k-1}) + C \sum_{v \in V_k} \alpha_v^2 2^{-k},$$

since this estimate could be iterated k times to produce (9.1). Before getting down on the actual details, I briefly discuss why this **should** work, and why it actually **does not** quite work. Let $v \in V_k$. The basic (and slightly naive) idea is estimate

$$\mathcal{H}^1(\Gamma_v) \leq \mathcal{H}^1(\Gamma_{k-1} \cap \mathcal{N}(v)) + C\alpha_v^2 2^{-k}. \tag{9.2}$$

If $\alpha_v \geq \epsilon$, this is trivially true (with implicit constants depending on ϵ of course), since it is easy to check that $\mathcal{H}^1(\Gamma_v) \lesssim 2^{-k}$. If $\alpha_v < \epsilon$, the situation might look like the one in Figure 10. In particular, the vertices of Γ_k are ordered linearly relative to ℓ_v. Now, if cheating and over-optimism are allowed for a moment, the length of all Γ_k inside $\mathcal{N}(v)$ can be estimated as follows. First, for every edge $[v', v''] \subset \Gamma_k$ with $v', v'' \in V_k \cap \mathcal{N}(v)$, use Lemma 7.6 to infer that

$$\mathcal{H}^1([v', v'']) \leq \mathcal{H}^1([\pi_{\ell_v}(v'), \pi_{\ell_v}(v'')]) + C\alpha_v^2 2^{-k}.$$

Let's assume for simplicity that $\Gamma_k \cap \mathcal{N}(v)$ consists of edges only. Then, summing over the edges, and observing that the projection π_{ℓ_v} is injective on Γ_k, one arrives (roughly) at

$$\mathcal{H}^1(\Gamma_v) \leq \mathcal{H}^1(\ell_v \cap \mathcal{N}(v)) + C\alpha_v^2 2^{-k}.$$

There are several cheats here, so do not take the estimate above seriously! Finally, if Γ_{k-1} "spans through the entire ball $\mathcal{N}(v)$", meaning that every point $t \in \ell_v \cap \mathcal{N}(v)$ can be obtained as a projection $\pi_{\ell_v}(x)$ for some $x \in \Gamma_{k-1} \cap \mathcal{N}(v)$, then

$$\mathcal{H}^1(\Gamma_v) \leq \mathcal{H}^1(\ell_v \cap \mathcal{N}(v)) + C\alpha_v^2 2^{-k} \leq \mathcal{H}^1(\Gamma_{k-1} \cap \mathcal{N}(v)) + C\alpha_v^2 2^{-k},$$

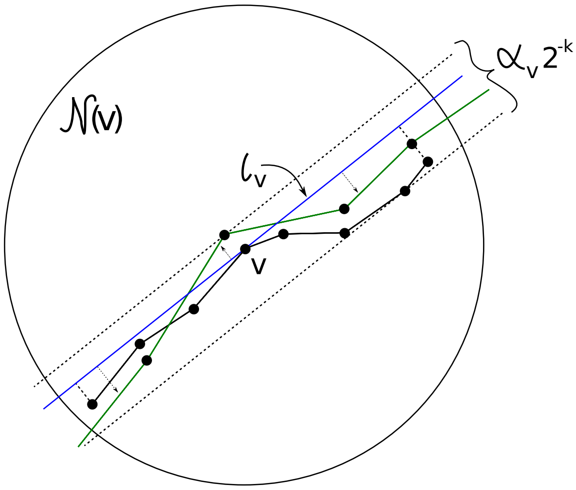

FIGURE 10. The set Γ_k is drawn in black, and the part of Γ_{k-1} inside $\mathcal{N}(v)$ is drawn in green. The line ℓ_v, which now coincidentally happens to pass through v, is drawn in blue.

simply because π_{ℓ_v} is a 1-Lipschitz mapping and does not increase length. This is precisely the estimate we were after. Of course, in real life there may be bridges contained in Γ_v, and there will be problems near the boundary of $\mathcal{N}(v)$ (where edges are no longer spanned by two vertices in $V_k \cap \mathcal{N}(v)$). These problems are reason for serious headache, but they are little compared to the following big issue: even if Γ_v has non-trivial length, the set $\Gamma_{k-1} \cap \mathcal{N}(v)$ can be absolutely tiny, in fact a point! This can happen, for instance, if v is a vertex on a bridge and has many neighbours in $V_k \cap \mathcal{N}(v)$, but $w = v^{\uparrow}$ has no neighbours in $V_{k-1} \cap \mathcal{N}(v)$, see Figure 11.

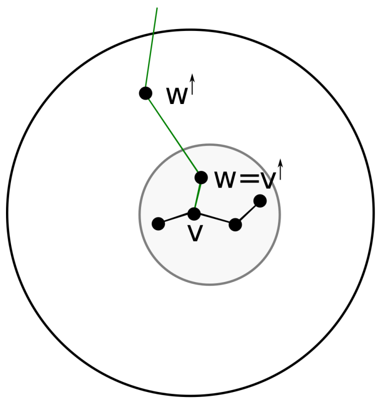

FIGURE 11. The vertex v belongs to the same bridge as $w = v^{\uparrow} \in V_{k-1}$ and $w^{\uparrow} \in V_{k-2}$. The vertex w is the only vertex of V_{k-1} inside $\mathcal{N}(v)$; in particular, $w = (v')^{\uparrow}$ for all the vertices $v' \in V_k \cap \mathcal{N}(v)$, which forces these vertices to lie quite close to w – and each other. Thus, these vertices are inter-connected by edges in Γ_k.

Clearly, there is **no** chance for anything like (9.2) to work in this scenario. This is the main problem in the proof, and it is resolved with a very clever "pre-payment" scheme. This is formalised through the notion of *virtual credit*, which we now start to discuss.

9.1. **Virtual credit.** The *virtual credit* associated with $v \in V_k$ is the number

$$\$_v = \$_{v,k} = \$_k := 3A2^{-k}.$$

The virtual credit of a bridge $B(v,v')$, $v,v' \in V_k$ is

$$\$_{v,v'} := \sum_{w \in B(v,v')} \$_w = 2 \cdot 3A \sum_{j \geq k} 2^{-k} = 12^{-k}.$$

Virtual credit is an allegory of life itself: it comes and goes, not everyone has it all the time, and everyone loses it in the end. For every $k \geq 0$, we will inductively define a set

$$R_k \subset \bigcup_{l \geq k} V_l,$$

whose elements are "rich", and have credit at time k. Other elements are poor and have nothing. The total virtual credit at time k equals

$$\$(R_k) := \sum_{v \in R_k} p_v.$$

The initial set R_0 of rich vertices consists of all the elements of $\bigcup_{k \geq 0} V_k$, which make an appearance in the definition of Γ_0. In other words, every $v \in V_0$ lies in R_0, and also $B(v,v') \subset R_0$ for all $v,v' \in V_0$. To get the induction rolling, I state two key properties, which will always be required from R_k:

(BP) The "bridge property" states that whenever $B(v,v') \subset \Gamma_k$ with $v,v' \in V_j$ and $0 \leq j \leq k$, then R_k contains all vertices in $B(v,v') \cap \bigcup_{l \geq k+1} V_l$.

(TVP) The "terminal vertex property" is a bit more complicated, but it essentially states that "those without neighbours are rich". To be precise, fix $v \in V_k$, and let ℓ be any line such that

$$\operatorname{dist}(y,\ell) < \epsilon \cdot 2^{-k}, \qquad y \in V_k \cap B(v, 30A2^{-k}).$$

Then the points in $V_k \cap BN(v, 30A2^{-k})$ are arranged so that "left" and "right" make sense. In case there is no vertex of V_k **either to the left or to the right** of v inside $B(v, 30A2^{-k})$, then $v \in R_k$.

These properties are trivially satisfies by R_0. So, next we assume that R_{k-1} has already been defined for some $k \geq 1$, satisfying the properties (BP) and (TVP). Let us see, how to define R_k. First, initialise R_k by setting

$$R_k := R_{k-1} \setminus [V_{k-1} \cup V_k].$$

Next, we consider each element $v \in V_k$ and add vertices to R_k according to the familiar cases (L), (S) etc.

(L) If $\alpha_v \geq \epsilon$, then all vertices $v' \in V_k \cap \mathcal{N}(v)$ are added to R_k. Also, if $B(v',v'') \subset \Gamma_{k,v}$, then $B(v',v'') \subset R_k$.

(S-NT) If $\alpha_v < \epsilon$, and **both** Γ_v^R **and** Γ_v^L were defined via Case (S-NT), then no vertices are added to R_k. Thus, v is terminal to neither left nor right.

(S-TT) If $\alpha_v < \epsilon$ and **either** Γ_v^R **or** Γ_v^L was defined via Case (S-TT), then v is added to R_k.

(S-TB) If $\alpha_v < \epsilon$ and Γ_v^R was defined via Case (S-TB), add $B(v, v_1)$ to R_k. Similarly, if Γ_v^L was defined via Case (S-TB), then add $B(v_{-1}, v)$ to R_k.

This completes the definition of R_k, and it is clear that R_k satisfies the bridge property (BP) (note that if $v \in R_{k-1} \cap V_{k+1}$, then also $v \in R_k \cap V_{k+1}$). It is also clear (by induction) that $R_k \subset \bigcup_{l \geq k} V_l$. It remains to verify that R_k satisfies the terminal vertex property (TVP). So, fix $v \in V_k$ and let ℓ be any such line that

$$\operatorname{dist}(y, \ell) < \epsilon \cdot 2^{-k}, \qquad y \in V_k \cap B(v, 30A2^{-k}). \tag{9.3}$$

Assume there is no vertex either to the "left" or "right" of v in the ordering of $V_k \cap B(v, 30A2^{-k})$ with respect to ℓ. Since ℓ is a completely arbitrary line, the estimate (9.3) tells us nothing about ℓ_v or α_v: in particular, it could happen that $\alpha_v \geq \epsilon$. But in this case $v \in R_k$ by item (L) above, so we are happy. So, assume $\alpha_v < \epsilon$. Then the set $V_k \cap B(v, 30A2^{-k})$ is also ordered relative to ℓ_v, and, by choosing orientations correctly, these orderings agree by Lemma 7.6. Thus, the fact that there is no vertex to the "left" or "right" from v means that v is either terminal to the left or right, and hence one of the items (S-TT) or (S-TB) occur. In both cases $v \in R_k$, and the induction is complete.

9.2. **Proof of the length estimate (9.1).** Now we are finally set to prove the estimate (9.1), which is repeated below:

$$\mathcal{H}^1(\Gamma_k) \lesssim 1 + \sum_{j \leq k} \sum_{v \in V_j} \alpha_v^2 2^{-j}. \tag{9.4}$$

The key auxiliary estimate is the following. Let Edges(k) be the edges $[v, v'] \subset \Gamma_k$, and let Bridges(k) be the bridges $B(v, v') \subset \Gamma_k$ with $v, v' \in V_k$ (note that Γ_k may contain other bridges than those included in the "generation k bridges" Bridges(k)). Then

$$\begin{aligned} \sum_{[v,v'] \in \text{Edges}(k)} \mathcal{H}^1([v, v']) &+ \sum_{B(v,v') \in \text{Bridges}(k)} \mathcal{H}^1(B(v, v')) + \$(R_k) \\ &\leq \sum_{[w,w'] \in \text{Edges}(k-1)} \mathcal{H}^1([w, w']) + \frac{13}{15} \sum_{B(v,v') \in \text{Bridges}(k)} \mathcal{H}^1([v, v']) \\ &\quad + \$(R_{k-1}) + C \sum_{v \in V_k} \alpha_v^2 2^{-k}. \end{aligned} \tag{9.5}$$

The reader might first think that there are typos on line (9.5), but there are none: the sum should **not** run over the bridges of generation $k - 1$, and we really want to sum over $\mathcal{H}^1([v, v'])$ instead of $\mathcal{H}^1(B(v, v'))$.

9.2.1. *Proof of* (9.4) *based on* (9.5). By definition of Γ_k,

$$\mathcal{H}^1(\Gamma_k) \leq \sum_{[v,v'] \in \text{Edges}(k)} + \sum_{j \leq k} \sum_{B(w,w') \in \text{Bridges}(j)} \mathcal{H}^1(B(w, w')),$$

and hence (9.5) leads to

$$\begin{aligned}\mathcal{H}^1(\Gamma_k) &+ \$(R_k)\\ &\leq \sum_{[w,w']\in \mathrm{Edges}(k-1)} \mathcal{H}^1([w,w']) + \sum_{j\leq k-1} \sum_{B(w,w')\in \mathrm{Bridges}(j)} \mathcal{H}^1(B(w,w')) + \$(R_{k-1})\\ &\quad + \frac{13}{15} \sum_{B(v,v')\in \mathrm{Bridges}(k)} \mathcal{H}^1([v,v']) + C\sum_{v\in V_k} \alpha_v^2 2^{-k}\\ &= \mathcal{H}^1(\Gamma_{k-1}) + \$(R_{k-1}) + \frac{13}{15} \sum_{B(v,v')\in \mathrm{Bridges}(k)} \mathcal{H}^1([v,v']) + C\sum_{v\in V_k} \alpha_v^2 2^{-k}.\end{aligned}$$

Now, performing the same estimate on $\mathcal{H}^1(\Gamma_{k-1}) + \$(R_{k-1})$, and continuing in the same manner k times, leads to

$$\mathcal{H}^1(\Gamma_k) \leq \mathcal{H}^1(\Gamma_0) + \$(R_0) + \frac{13}{15}\sum_{j\leq k} \sum_{B(v,v')\in \mathrm{Bridges}(j)} \mathcal{H}^1([v,v']) + C\sum_{j\leq k}\sum_{v\in V_j} \alpha_v^2 2^{-j}. \quad (9.6)$$

To conclude (9.4) from here, note that $[v,v'] \subset B(v,v') \subset \Gamma_k$ for all $B(v,v') \in \mathrm{Bridges}(j)$ and for all $j \leq k$. Moreover, the sets $[v,v']$ arising this way are essentially disjoint, meaning that

$$\mathcal{H}^1([v,v'] \cap [w',v'']) = 0.$$

for distinct pairs v,v' and w,w'. This follows immediately from the initial reduction we made in Proposition 7.4. Consequently,

$$\frac{13}{15}\sum_{j\leq k} \sum_{B(v,v')\in \mathrm{Bridges}(j)} \mathcal{H}^1([v,v']) \leq \frac{13}{15}\mathcal{H}^1(\Gamma_k).$$

Now (9.4) follows from (9.6), combined with the nearly trivial estimate

$$\mathcal{H}^1(\Gamma_0) + \$(R_0) \lesssim 1.$$

9.3. **Proof of the estimate (9.5).** I repeat the estimate below:

$$\begin{aligned}\sum_{[v,v']\in \mathrm{Edges}(k)} \mathcal{H}^1([v,v']) &+ \sum_{B(v,v')\in \mathrm{Bridges}(k)} \mathcal{H}^1(B(v,v')) + \$(R_k)\\ &\leq \sum_{[w,w']\in \mathrm{Edges}(k-1)} \mathcal{H}^1([w,w']) + \frac{13}{15} \sum_{B(v,v')\in \mathrm{Bridges}(k)} \mathcal{H}^1([v,v']) \qquad (9.7)\\ &\quad + \$(R_{k-1}) + C\sum_{v\in V_k} \alpha_v^2 2^{-k}.\end{aligned}$$

Staring at the left hand side for a moment reveals that it can be split into "local" terms of the form

$$\sigma(v) := \sum_{[v'v'']\in \mathrm{Edges}(k,v)} \mathcal{H}^1([v',v'']) + \sum_{B(v',v'')\in \mathrm{Bridges}(k,v)} \mathcal{H}^1(B(v',v'')) + \$(R_k(v)), \quad v \in V_k,$$

where $\mathrm{Edges}(k,v)$ and $\mathrm{Bridges}(k,v)$ stand for the bridges and edges added to Γ_v, and $R_k(v)$ is the part of R_k constructed with $v \in V_k$ fixed (recall the definition of R_k). However, simply estimating the left hand side of (9.7) by a sum of the local terms $\sigma(v)$ over $v \in V_k$ is wasteful: for instance, each edge $[v',v''] \subset \Gamma_k$ only needs to be counted once, even if it may (and most often will) be included in Γ_v for several distinct vertices v. Of course, even a wasteful estimate can sometimes work, but here it is too rough. Namely, every term on the right hand side of (9.7) can also be used only **once** to "pay" for something on the left hand side, and this is essentially why terms on the left hand side should also be accounted for precisely once.

At a high level, proving the estimate (9.7) thus has two challenges: first, to estimate each term $\sigma(v)$ separately by something appearing on the right hand side of (9.7), and, second, to make sure that nothing on the right hand side gets used twice in such estimates, when $v \in V_k$ varies. Thus, the proof will (formally speaking) contain the construction of an injective mapping Ψ from all the terms on the left hand side to those on the right hand side. I will never attempt to write the complete expression of Ψ down, but this philosophy is good to keep in mind.

The proof now begins. The local terms $\sigma(v)$ with $\alpha_v \geq \epsilon$ allow for a very care-free estimate:

9.3.1. *Edges, bridges and virtual credit nearby a case (L) vertex.* Assume that $\alpha_v \geq \epsilon$. Then

$$\sigma(v) \lesssim 2^{-k} \lesssim \alpha_v^2 2^{-k},$$

which is certainly good enough for us. The other cases will be more involved, but we can already **benefit** from the fact that Case (L) has been settled: in case an edge, or bridge, or virtual credit appearing below **also** happens to appear in some local term $\sigma(v)$ with $\alpha_v \geq \epsilon$, then we know that this edge/bridge/virtual credit has already been accounted for, and can be ignored.

9.3.2. *Virtual credit and parts of edges very close to Case* (S-TT) *vertices.* Assume that $\alpha_v < \epsilon$, so that both $V_k \cap \mathcal{N}(v)$ and $V_{k-1} \cap \mathcal{N}(v)$ are ordered along the line $\ell = \ell_v$, and "left" and "right" make sense. In this subsection, we will not handle the full sum $\sigma(v)$ for a Case (S-TT) vertex $v \in V_k$, but only a part of it. The rest will come later. For the moment, we are indeed just interested in bounding the quantity

$$\$_v + \sum_{[v',v''] \in \mathrm{Edges}(k,v)} \mathcal{H}^1([v',v''] \cap B(v, 2A2^{-k})) \tag{9.8}$$

for a fixed vertex $v \in V_k$, for which **either** Γ_v^L **or** Γ_v^R was defined through Case (S-TT). Note that in this case $v \in R_k$, so $\$_v$ is indeed a part of $\$(R_k)$ and appears on the left hand side of (9.7). The quantity in (9.8) will be bounded by either $\$_w$ or $\$_w + \$_{w^\downarrow}$ for certain $w, w^\downarrow \in R_{k-1}$ (which are, in turn, terms appearing on the right hand side of (9.7)).

There are a few cases to consider. You probably need to recall what Case (S-TT) means: in particular, recall the definition of the left-most and right-most vertices

$$w_l \in V_{k-1} \cap B(v, 2A2^{-k}) \quad \text{and} \quad w_r \in V_{k-1} \cap B(v, 2A2^{-k}).$$

Observe that certainly

$$\mathrm{dist}(y, \ell) < \epsilon \cdot 2^{-(k-1)}, \qquad y \in V_{k-1} \cap B(w, 30A2^{-(k-1)}), \qquad w \in \{w_l, w_w\}, \tag{9.9}$$

because $B(w, 30A2^{-(k-1)}) \subset B(v, 65A2^{-k})$, and using the definition of $\alpha_v < \epsilon$. Now, assume that Γ_v^R, for instance, was defined through Case (S-TT). By definition, this means that there is no vertex of V_{k-1} "to the right" from w_r within $B(w, 30A2^{-(k-1)})$, and by the terminal vertex property (TVP), we conclude that $w_r \in R_{k-1}$. Now, once more using the fact that $\alpha_v < \epsilon$, recalling the length estimate in Lemma 7.6, and observing that there are no edges passing in the "right half" of $B(v, 2A2^{-k})$ (this is precisely because Γ_v^R is defined via Case (S-TT), see Figure 12), one obtains

$$\$_v + \sum_{[v',v'']\in \text{Edges}(k,v)} \mathcal{H}^1([v',v'']\cap B(v,2A2^{-k})) = 3A2^{-k} + (1+3\epsilon^2)2A2^{-k} < 3A2^{-(k-1)} = \$_{w_r},$$

Of course, the estimate above would hold even with $w_r \notin R_{k-1}$, but the point is that now $\$_{w_r}$ is something appearing on the right hand side of (9.7). In case Γ_v^L was defined via Case (S-TT), the estimate is the same, with w_r replaced by w_l, and "left" and "right" interchanged.

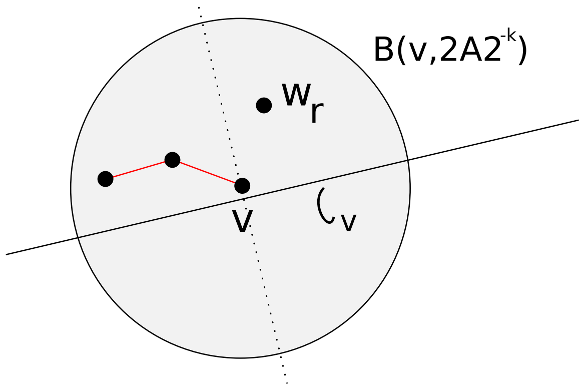

FIGURE 12. The task is to estimate the length of the red curve.

Are we done? **No!** It might occur that a single vertex $w \in R_{k-1}$ is needed in the treatment of many distinct Case (S-TT) vertices v as above (so the injectivity of the mapping Ψ is in jeopardy). More precisely, there could be **several** vertices v such that either

- $w = w_r^v$, where Γ_v^R was defined via Case (S-TT), or
- $w = w_l^v$, where Γ_v^L was defined via Case (S-TT).

I emphasise again that this is a real problem, because $\$_w$ may only be counted once in $\$(R_{k-1})$ on the right hand side of (9.7). It turns out that **several** can be exactly **twice** (as we will shortly see), but even so $\$_w$ is not large enough alone. To start tackling this problem, first note that whenever w arises as w_r^v or w_l^v for some such vertex v, then $w \in B(v, 2A2^{-k})$, so $v \in B(w, 2A2^{-k})$.

Now, assume that there are at least two Case (S-TT) vertices $v_1, v_2 \in V_k \cap B(w, 2A2^{-k})$, which give rise to the same w in the argument above. Then, inside the ball $B(w, 40A2^{-k})$ (this is just some ball large enough to contain all the interesting action) the vertices of V_k are ordered with respect to $\ell = \ell_{v_1}$, say, and "left" and "right" make sense. Assume that v_1 is "left" from v_2: then v_1 is terminal to the left, and v_2 is terminal to the right. Since both are Case (S-TT) vertices, the conclusion is that $\Gamma_{v_1}^L$ must have been defined via Case (S-TT), and $\Gamma_{v_2}^R$ must have been defined via Case (S-TT). Now, we are in trouble, if

$$w = w_l^{v_1} \quad \text{and} \quad w = w_r^{v_2}.$$

If this is really the case, then the fact that $\Gamma_{v_1}^L$ was defined via Case (S-TT) implies that there are no vertices of V_{k-1} within distance $30A2^{-(k-1)}$ to the left from $w_l^{v_1} = w$. Similarly, there are no vertices of V_{k-1} within distance $30A2^{-(k-1)}$ to the right from w. So, w

is in fact the only vertex in $V_{k-1} \cap B(w, 30A2^{-(k-1)})$, which forces

$$v_1^{\uparrow} = w = v_2^{\uparrow},$$

This gives the slightly improved estimate

$$|v_1 - v_2| \leq |v_1 - w| + |w - v_2| \leq 2A2^{-k}.$$

In particular,

$$L := \sum_{[v',v'']\in \mathrm{Edges}(k)} \mathcal{H}^1([v',v''] \cap (B(v_1, 2A2^{-k}) \cup B(v_2, 2A2^{-k})) \leq (1+3\epsilon^2)2A2^{-k} < 3A2^{-k} = \frac{\$_w}{2},$$

using Lemma 7.6 again, and noting that all the possible edges in the summation must lie between v_1 and v_2 (in the ordering with respect to ℓ). Moreover, possible vertices v (strictly) between v_1 and v_2 **cannot** give rise to w in the sense that v_1 and v_2 do, because they are, evidently, not terminal in either direction. This is why **at most** the two named vertices v_1, v_2 can give rise to w.

The final observation is that since w is not connected by an edge to any other vertices in V_{k-1} (all such vertices are too far away, at distance $30A2^{-(k-1)}$ at least), but since w is still **in some manner** connected to other vertices in Γ_{k-1}, it must be the case that w belongs to a bridge $B(x,y)$ for some $x, y \in V_j$, $j < k-1$. Thus also $w^{\downarrow} \in B(x,y) \cap V_k$, and hence $w^{\downarrow} \in R_{k-1}$ by the bridge property (BP) of virtual credit. Since

$$\$_{v_1} + \$_{v_2} = \$_{w^{\downarrow}} + \frac{\$_w}{2},$$

we now arrive at the estimate

$$L + \$_{v_1} + \$_{v_2} \leq \$_w + \$_{w^{\downarrow}}.$$

This means that even in the worst case, when w is needed **twice**, every element of R_{k-1} is only needed **once** in (this part of) the estimate (9.7)!

9.3.3. *Virtual credit, edges and bridges near Case (S-TB) vertices.* Suppose that $v \in V_k$ has $\alpha_v < \epsilon$, so that both $V_k \cap \mathcal{N}(v)$ and $V_{k-1} \cap \mathcal{N}(v)$ are ordered along the line $\ell = \ell_v$, and "left" and "right" make sense. In this subsection, we assume that at least one of Γ_v^R and Γ_v^L was defined via Case (S-TB). For instance, assume that Γ_v^R was defined via Case (S-TB), so that v_1, the "next vertex to the right" from v inside $\mathcal{N}(v)$ exists, and lies at distance $\geq 30A2^{-k}$ from v, see Figure 13. This time we will handle the following part of $\sigma(v) + \sigma(v_1)$:

$$\$_{v,v_1} + \sum_{[v',v'']\in \mathrm{Edges}(k)} \mathcal{H}^1([v',v''] \cap B(\{v, v_1\}, 2A2^{-k})) + \mathcal{H}^1(B(v, v_1)). \tag{9.10}$$

Here $B(\{v, v_1\}, 2A2^{-k}) := B(v, 2A2^{-k}) \cup B(v_1, 2A2^{-k})$. We will bound (9.10) by

$$\mathcal{H}^1([w_r, w_{r+1}]) + \frac{13}{15}\mathcal{H}^1([v, v_1]).$$

Note that both these terms above appear on the right hand side of (9.7), and we have not used them in the previous cases, so we are free to waste them here. Start by observing

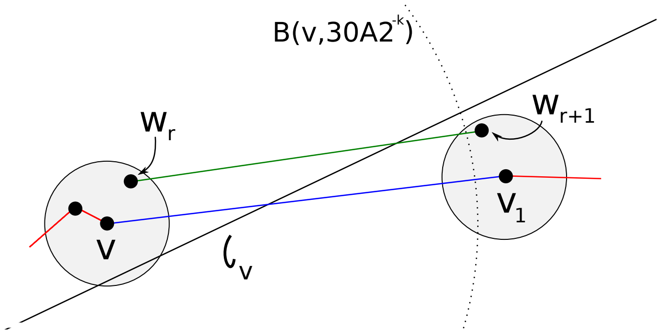

FIGURE 13. The possible location of the edges in $\Gamma_k \cap B(\{v, v_1\}, 2A2^{-k})$ are marked in red, a part of the bridge in Γ_k is marked in blue, and the edge $[w_r, w_{r+1}] \subset \Gamma_{k-1}$ is marked in green. The fact that $w_{r+1} \in B(v_1, 2A2^{-k})$ uses (7.10).

that

$$\mathcal{H}^1(B(v, v_1)) \leq \mathcal{H}^1([v, v_1]) + 2\sum_{j=0}^{\infty} A2^{-j} = \mathcal{H}^1([v, v_1]) + 4A2^{-k},$$

by repeated application of the hypothesis $(V^{\downarrow})$. Also, recall that $\$_{v,v_1} = 12A2^{-k}$. The lengths of the edges inside $B(v, 2A2^{-k})$ can be estimated by

$$\sum_{[v',v'']\subset \mathrm{Edges}(k)} \mathcal{H}^1([v', v'] \cap B(v, 2A2^{-k})) \leq (1 + 3\epsilon^2)2A2^{-k} < 3A2^{-k}$$

by Lemma 7.6, and the same holds for the edges inside $B(v_1, 2A2^{-k})$. All in all, (9.10) turns out to be at most

$$(4 + 12 + 6)A2^{-k} + \mathcal{H}^1([v, v_1]) = 22A2^{-k} + \mathcal{H}^1([v, v_1]).$$

Next, note that

$$\mathcal{H}^1([w_r, w_{r+1}]) \geq \mathcal{H}^1([v, v_1]) - 4A2^{-k},$$

because $w_r \in B(v, 2A2^{-k})$ and $v_1 \in B(w_{r+1}, 2A2^{-k})$ by (7.10). This proves that

$$(9.10) \leq \mathcal{H}^1([w_r, w_{r+1}]) + 26A2^{-k} \leq \mathcal{H}^1([w_r, w_{r+1}]) + \frac{13}{15}\mathcal{H}^1([v, v_1]),$$

using the assumption $|v - v_1| \geq 30A2^{-k}$.

Again, it is a legitimate concern, whether this case is now really complete: could it, again, happen that a term of the form $\mathcal{H}^1([w_r, w_{r+1}])$ or $\frac{13}{15}\mathcal{H}^1([v, v_1])$ comes up several times, as one varies the point v? This does happen, indeed, but only twice: for v itself, and then v_1, and this is precisely why these two terms received a symmetrical treatment above. You should now think, how the sum (9.10) had looked like, had we started off with v_1 instead of v. If $\alpha_{v_1} < \epsilon$, then you will find that $\Gamma^L_{v_1}$ is defined via Case (S-TB), and v is the first vertex to the left from v_1. Consequently, (9.10) looks the same for v and v_1. But since these terms of (9.10) only need to be counted once in the sum (9.7), this is ok.

Now, we prove that the terms $\mathcal{H}^1([w_r, w_{r+1}])$ or $\frac{13}{15}\mathcal{H}^1([v, v_1])$ can only arise from this case for v or v_1. So, assume that $v' \neq v$ is another vertex with this property, and let us prove that $v' = v_1$. Since v' is relevant for this case, v' must have either a left or a right neighbour v'_{-1} or v'_1 such that $B(v'_{-1}, v') \subset \Gamma_k$ or $B(v', v'_1) \subset \Gamma_k$. Assume, say, that v' has such a left neighbour v'_{-1}.

Now, the proof above applied to the pair v', v'_{-1} gives rise to the terms

$$\tfrac{13}{15}\mathcal{H}^1([v'_{-1}, v']) \quad \text{and} \quad \mathcal{H}^1([w'_{l-1}, w'_l]),$$

which are needed in estimating the analogue of (9.10). We potentially run into trouble, if either $[v'_{-1}, v'] = [v, v_1]$ or $[w'_{l-1}, w'_r] = [w_r, w_{r+1}]$, because the terms $\mathcal{H}^1([v, v_1])$ and $\mathcal{H}^1([w_r, w_{r+1}])$ are needed (also) in connection with v. In case $[v'_{-1}, v'] = [v, v_1]$, then $v' = v_1$, as claimed.

What if $[w'_{l-1}, w'_r] = [w_r, w_{r+1}]$, so that $w'_{l-1} = w_r =: w_1$ and $w'_r = w_{r+1} =: w_2$? Then

$$v, v'_{-1} \in B(w_1, 2A2^{-k}) \quad \text{and} \quad v_1, v' \in B(w_2, 2A2^{-k}),$$

But now all the points v, v', v'_{-1}, v_1 are linearly ordered with respect to ℓ_v, say, and this easily implies $v_1 = v'$. Otherwise v_1 would either be strictly to the left or right from v'. If left, then v'_{-1} certainly would not be the nearest point left from v'. If right, then v_1 certainly would not be the nearest point right from v. This proves that $v_1 = v'$.

We soon move to the last case: it will be crucial to keep in mind that certain edges $[w, w'] \in \text{Edges}(k-1)$ have already been used in the current case. So, the reader needs to make sure that those edges do not get used again!

9.3.4. *Whatever remains.* What actually remains? It is probably a good idea to have a look at (9.7) once more:

$$\begin{aligned}
&\sum_{[v,v'] \in \text{Edges}(k)} \mathcal{H}^1([v, v']) + \sum_{B(v,v') \in \text{Bridges}(k)} \mathcal{H}^1(B(v, v')) + \$(R_k) \\
&\quad \leq \sum_{[w,w'] \in \text{Edges}(k-1)} \mathcal{H}^1([w, w']) + \frac{13}{15} \sum_{B(v,v') \in \text{Bridges}(k)} \mathcal{H}^1([v, v']) \qquad (9.11) \\
&\qquad + \$(R_{k-1}) + C \sum_{v \in V_k} \alpha_v^2 2^{-k}.
\end{aligned}$$

It is clear that $\mathcal{H}^1(B(v, v'))$ has been dealt with for every bridge $B(v, v') \in \text{Bridges}(k)$ by the previous case. It is **not** true that every element of R_k has been taken care of: we have only accounted for the new additions to R_k at step k (let us denote them by N_k), and a quick look at Section 9.3.2 reveals that we have done so by using exclusively the virtual credit in $R_{k-1} \cap [V_{k-1} \cup V_k]$. But

$$R_k \setminus N_k \subset R_{k-1} \setminus [V_{k-1} \cup V_k],$$

because R_k was initialised by deleting everything from $R_{k-1} \cap [V_{k-1} \cup V_k]$, so

$$\$(R_k \setminus N_k) \leq \$(R_{k-1} \setminus [V_{k-1} \cup V_k]).$$

This implies that $\$(R_k)$ is now completely accounted for. Consequently, all that remains "to be paid for" on the left hand side of (9.11) are certain edges, and parts thereof. More precisely, in Sections 9.3.1–9.3.3, we have already taken care of edges, and parts thereof, which are contained in either $\mathcal{N}(v)$ for some $v \in V_k$ with $\alpha_v \geq \epsilon$, or alternately $B(v, 2A2^{-k})$ for some vertex $v \in V_k$, which is terminal to either left or right.

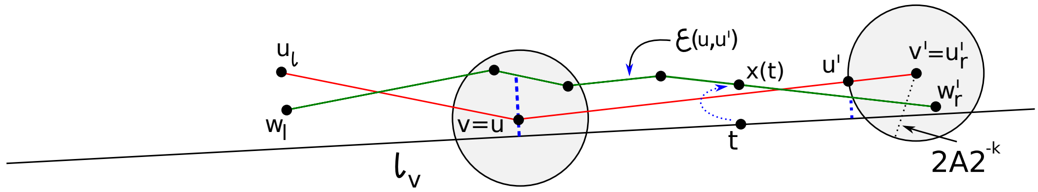

FIGURE 14. The position of the vertices $v, v', u, v', u_l, u_r', w_l$ and w_r'. In this scenario v' happens to be a terminal vertex, whereas v is a non-terminal vertex. The set Γ_k is drawn in red, and the set Γ_{k-1} is drawn in green. The set $\mathcal{E}(u,u')$ is the part of Γ_{k-1} between the dotted blue lines.

Now, fix an edge $[v,v'] \subset \Gamma_k$ with $v,v' \in V_k$, and such that $\max\{\alpha_v, \alpha_{v'}\} < \epsilon$. Let $[u,u'] \subset [v,v']$ be (any) maximal sub-segment, which stays at distance $\geq 2A2^{-k}$ from **all** terminal vertices. Let π be the orthogonal projection to the line ℓ_v. By Lemma 7.6,

$$\mathcal{H}^1([u,u']) \leq (1+3\alpha_v^2)\mathcal{H}^1([\pi(u),\pi(u')]) \leq \mathcal{H}^1([\pi(u),\pi(u')]) + 90A\alpha_v^2 2^{-k},$$

since π is 1-Lipschitz and $|u-u'| \leq 30A2^{-k}$ by the **Principle**. Assume, say, that u lies to the left form u' in the order relative to ℓ_v (this makes sense, as $\alpha_v < \epsilon$). Then, let $u_l \in V_k$ (resp. $u_r' \in V_k$) be the closest vertex to the left from u (resp. right from u') with

$$\pi(u_l) < \pi(u) - A2^{-k} \quad \text{and} \quad \pi(u_r') > \pi(u) + A2^{-k}.$$

Why do they exist? If, for instance, all vertices to the left from u within $\mathcal{N}(v)$ satisfied the opposite inequality, then all of them would certainly lie in $B(u, 2A2^{-k})$, and the leftmost of them would be terminal to the left, contrary to the definition of u. The situation is depicted in Figure 14. Finally, use hypothesis $(V^{\uparrow})$ to find vertices $w_l = u_l^{\uparrow}$ and $w_r' = (u_r')^{\uparrow}$ with $|w_l - u_l| < A2^{-k}$ and $|w_r' - u_r'| < A2^{-k}$. It follows that

$$\pi(w_l) < \pi(u) < \pi(u') < \pi(w_r'). \tag{9.12}$$

Moreover, the vertices w_l and w_r can be connected by a finite sequence of edges inside $\Gamma_{k-1} \cap \mathcal{N}(v)$, which essentially follows from the **Principle**: w_l is fairly close to v (at distance $\leq 32A2^{-k} \ll 30A2^{-(k-1)}$), so w_l and $v^{\uparrow}$ can first be connected by edges in Γ_{k-1} inside $\mathcal{N}(v)$. Then, $v^{\uparrow}$ can be connected to w_r', since w_r' is not much further from $v^{\uparrow}$ than $|v-v'| \leq 30A2^{-k} \ll 30A2^{-(k-1)}$.

Now, it follows from (9.12) and the discussion above that for every point $t \in [\pi(u), \pi(u')]$, there is a point $x(t) \in \Gamma_{k-1}$, belonging to one of the edges connecting w_l to w_r', such that $\pi(x(t)) = t$ (see Figure 14). Moreover, this point x can be chosen so that

$$|x(t) - t| \leq \alpha_v 2^{-k}. \tag{9.13}$$

(Indeed, since the end-points of all edges $[w,w']$ fully contained $\Gamma_{k-1} \cap \mathcal{N}(v)$ satisfy $\operatorname{dist}(w, \ell_v) < \alpha_v 2^{-k}$, the same distance bound remains true for any points on $[w,w']$.) Consequently, using again the fact that π is 1-Lipschitz,

$$\mathcal{H}^1([\pi(u),\pi(u')]) \leq \mathcal{H}^1(\mathcal{E}(u,u')),$$

where $\mathcal{E}(u,u') := \{(x(t) \in \Gamma_{k-1} : t \in [u,u']\}$. All in all,

$$\mathcal{H}^1([u,u']) \leq \mathcal{H}^1(\mathcal{E}(u,u')) + 90\alpha_v^2 2^{-k}, \tag{9.14}$$

where $\mathcal{H}^1(\mathcal{E}(u,u'))$ is certainly a part of the sum

$$\sum_{[w,w']\in \text{Edges}(k-1)} \mathcal{H}^1([w,w']).$$

It is also easy to see that those edges $[w,w']$ do appear in this manner, which arose (and whose length was already used) in the previous section. Any such edge $[w,w']$ had the property that both w and w' lay at distance $\leq 2A2^{-k}$ from Case (S-TB) vertices in V_k, and there are none of those close enough.

So, the only remaining problem is that the sets $\mathcal{E}(u,u')$ can have some overlap as u,u' vary. Assume that two sets of the form $\mathcal{E}_1 := \mathcal{E}(u_1,u_1')$ and $\mathcal{E}_2 := \mathcal{E}(u_2,u_2')$ meet at a point $\xi \in \Gamma_{k-1}$, where $[u_1,u_1']$ and $[u_2,u_2']$ are distinct segments. The first task is to show that "all the action happens in a single local picture $\mathcal{N}(v)$". Here are some basic facts: $[u_1,u_1'] \subset [v_1,v_1']$ and $[u_2,u_2'] \subset [v_2,v_2']$ for certain vertices v_1,v_2,v_1',v_2' with α-numbers at most ϵ, satisfying

$$|v_1 - v_1'| < 30A2^{-k} \quad \text{and} \quad |v_2 - v_2'| < 30A2^{-k}.$$

Moreover, it is easy to check that $\mathcal{E}(u_1,u_1') \subset B(v_1, 32A2^{-k}) \cap B(v_1', 32A2^{-k})$ and similarly $\mathcal{E}(u_2,u_2') \subset B(v_2, 32A2^{-k}) \cap B(v_2', 32A2^{-k})$. In particular, the point ξ lies in all of the balls above. Now, it does not make a big difference, which vertex v_i or v_i' we declare as our "centre" v: say $v := v_1$. Then $\alpha_v < \epsilon$, and all the sets above lie well inside $\mathcal{N}(v)$ (check this; or if you are lazy, just assume that the constant "65" in the definition of $\mathcal{N}(v)$ is replaced by 10^{10} – it's precise value is totally irrelevant in future applications).

The points v_i and v_i', $i \in \{1,2\}$ are now linearly ordered relative to $\ell = \ell_v$. It cannot happen that $v_1 = v_1'$ and $v_2 = v_2'$, because the sets $\mathcal{E}(u,u')$ arising from a single edge $[v_1,v_1']$ are clearly disjoint (all those sets are defined the fixed projection $\pi = \pi_{v_1}$, so the points $x(t) \in \Gamma_{k-1}$ are distinct for disjoint intervals $[u_1,u_1'],[u_2,u_2'] \subset [v_1,v_1']$).

Now there are essentially two different possibilities: either all the points v_i, v_i' are distinct, or then, say $v_1' = v_2$ and the three points v_1, v_2, v_2' are consecutive in the linear order relative to ℓ. The first situation actually cannot occur, if ξ exists: this follows from the separation $|v_1' - v_2| \geq 2^{-k}$, and the fact that ϵ is so small, which implies that the projections π_{v_1} and π_{v_2} are nearly the same (as will be discussed carefully below). Checking the details is a bit tedious, but the situation is shown in Figure 15.

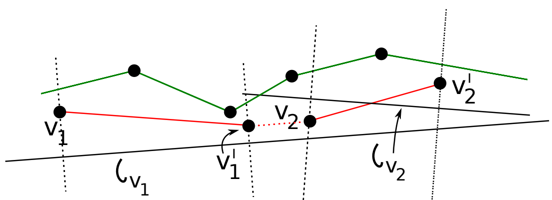

FIGURE 15. This is the case, where $v_1 < v_1' < v_2 < v_2'$. The edges in Γ_k are drawn in red, and the edges in Γ_{k-1} are shown in green. The parts of Γ_{k-1} required to pay for $[v_i, v_i']$ (or anything in between) are separated by the dotted lines. As you can see, these parts are distinct, because the separation between $|v_1' - v_2| \geq 2^{-k}$ is large compared to $\alpha_{v_1} 2^{-k}$.

Now, we are left with the case $v_1' = v_2$. I rename the three vertices $v_1, v_1' = v_2$ and v_2' as v_1, v_2, v_3, and the assumption is that $v_1 < v_2 < v_3$ are consecutive vertices in the linear order relative to ℓ_{v_1} (or any other ℓ_{v_i}, because these orders are compatible due to small α-numbers). This situation is depicted in Figure 16. This might be clear to the reader,

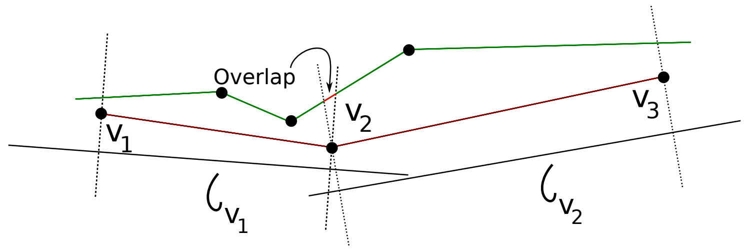

FIGURE 16. This is the case, where $v_1 < v_2 < v_3$, and moreover $[u_1, u_1'] = [v_1, v_2]$ and $[u_2, u_2'] = [v_2, v_3]$. The edges in Γ_k are drawn in deep red, and the edges in Γ_{k-1} are shown in green. The overlap $\mathcal{E}(v_1, v_2) \cap \mathcal{E}(v_2, v_3)$ is show in bright red.

but let us briefly repeat: what causes the overlap? The segments $[u_1, u_1']$ and $[u_2, u_2']$ are contained in the two distinct, consecutive segments $[v_1, v_2]$ and $[v_2, v_3]$. Now, recall the definition of $\mathcal{E}(u_i, u_i')$ from right above (9.14). For the segment $[u_i, u_i']$, $i \in \{1, 2\}$, it gives

$$\mathcal{E}(u_i, u_i') = \{x_{v_i}(t) : t \in [u_i, u_i']\},$$

where $x_{v_i}(t)$ is a point on Γ_{k-1}, close to t, such that $\pi_{v_i}(x(t)) = t$. Thus, the amount of overlap $\mathcal{E}(u_1, u_1') \cap \mathcal{E}(u_2, u_2')$ depends on how much the angles of the projections π_{v_1} and π_{v_2} differ from one another. If, for instance, $\alpha_{v_i} = 0$ for $i \in \{1, 2\}$, then all the points v_1, v_2, v_3 lie on both the lines ℓ_{v_i}, $i \in \{1, 2\}$, which forces the lines to coincide. In this case $\pi_{v_1} = \pi_{v_2}$, and there is, in fact, no overlap.

This suggests (rather optimistically) that the following estimate could hold:

$$\mathcal{H}^1(\mathcal{E}(u_1, u_1') \cap \mathcal{E}(u_2, u_2')) \lesssim \alpha^2 \cdot 2^{-k}, \qquad \alpha := \max\{\alpha_{v_1}, \alpha_{v_2}\}. \tag{9.15}$$

It turns out that (9.15) is true, as we will next verify. Note that this will complete the whole proof by (9.14).

To prove (9.15), let θ be the angle between the lines ℓ_{v_1} and ℓ_{v_2}. The overlap $\mathcal{E}(u_1, u_1') \cap \mathcal{E}(u_2, u_2')$ is then contained in a cone with opening angle θ and, by (9.13) at distance $\leq \alpha \cdot 2^{-k}$ from both of the segments

$$[\pi_{v_1}(v_1), \pi_{v_1}(v_2)] \subset \ell_{v_1} \quad \text{and} \quad [\pi_{v_2}(v_2), \pi_{v_2}(v_3)] \subset \ell_{v_2}.$$

It now suffices to show that $\theta \lesssim \alpha$, because then elementary geometry gives the estimate (9.14) (see Figure 17).

The estimate $\theta \lesssim \alpha$ is simple trigonometry. Consider the right-angled triangle (also shown in Figure 17) formed by the three points $\Delta_1 := \ell_{v_1} \cap \ell_{v_2}$, $\Delta_2 = \pi_{v_1}(v_3)$ and $\Delta_3 = \pi_{v_2}(\pi_{v_1}(v_3))$. Then the angle at Δ_1 is obviously θ, and so the sine of θ is

$$\sin\theta = \frac{|\Delta_2 - \Delta_3|}{|\Delta_2 - \Delta_1|} \lesssim \frac{\alpha \cdot 2^{-k}}{2^{-k}} = \alpha.$$

Hence $\theta \lesssim \alpha$, and the proof of Theorem 7.1 is complete.

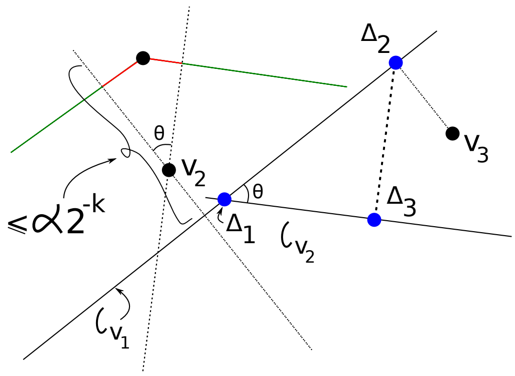

FIGURE 17. The overlap $\mathcal{E}(v_1, v_2) \cap \mathcal{E}(v_2, v_3)$ is shown in bright red. It's total length is clearly bounded by $\lesssim \theta \cdot \alpha 2^{-k}$. The triangle, from which θ can be solved, is marked by the three blue discs: the intersection of ℓ_{v_1} and ℓ_{v_1}, the projection $\pi_{v_1}(v_3)$, and the projection $\pi_{v_2}(\pi_{v_1}(v_3))$.

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UNIVERSITY OF HELSINKI, DEPT. OF MATHEMATICS AND STATISTICS
E-mail address: tuomas.orponen@helsinki.fi