GEOMETRIC MEASURE THEORY AND SINGULAR INTEGRALS: EXERCISE SET 1

In these exercises we work in $\mathbb{R}^d$ and $n \in (0, d]$.

(1) Suppose μ is a measure which is either finite or of order n (i.e. $\mu(B(x,r)) \le Cr^n$). Let K be a standard n-dimensional kernel. Show that for $\epsilon > 0$, $f \in \bigcup_{p\in[1,\infty)} L^p(\mu)$ and $x \in \mathbb{R}^d$ the integral
$$\int_{|x-y|>\epsilon} K(x,y) f(y)\, d\mu(y)$$
is absolutely convergent.

(2) Suppose μ is a measure of order n and K is a standard n-dimensional kernel. Suppose φ is a smooth function satisfying that $0 \le \varphi \le 1$, $\varphi = 0$ on $B(0, 1/2)$ and $\varphi = 1$ on $\mathbb{R}^d \setminus B(0,1)$. Define the smoothly truncated singular integrals
$$T_\epsilon^\varphi f(x) = \int K(x,y)\varphi\Big(\frac{|x-y|}{\epsilon}\Big) f(y)\, d\mu(y), \qquad \epsilon > 0.$$
Show that T_ϵ^φ, $\epsilon > 0$, are operators with standard n-dimensional kernels (with the kernel bounds being independent of ϵ).

(3) We continue with the setting of the previous exercise. Show by demonstrating a suitable pointwise bound that $(T_\epsilon)_{\epsilon>0}$ are uniformly bounded in $L^2(\mu)$ if and only if $(T_\epsilon^\varphi)_{\epsilon>0}$ are uniformly bounded in $L^2(\mu)$.

(4) Suppose μ is a locally finite measure on $\mathbb{R}^d$ and S is an operator acting on two complex measures $\nu_1, \nu_2 \in M(\mathbb{R}^d)$ that satisfies
$$\mu(\{x \in \mathbb{R}^d \colon\ |S(\nu_1,\nu_2)(x)| > \lambda\}) \lesssim \Big(\frac{\|\nu_1\|\|\nu_2\|}{\lambda}\Big)^{1/2}, \qquad \lambda > 0,\ \nu_1, \nu_2 \in M(\mathbb{R}^d).$$
Suppose H is a set with $\mu(H) \in (0,\infty)$. Show (by using Kolmogorov type arguments as in the proof of Cotlar's inequality) that
$$\Big(\frac{1}{\mu(H)}\int_H |S(\nu_1,\nu_2)|^{1/4}\, d\mu\Big)^4 \lesssim \frac{\|\nu_1\|}{\mu(H)}\frac{\|\nu_2\|}{\mu(H)}.$$

(5) Suppose n is an integer and μ is a measure satisfying $cr^n \le \mu(B(x,r)) \le Cr^n$ for all $x \in \operatorname{spt}\mu$ and $0 < r \le \operatorname{diam}(\operatorname{spt}\mu)$. For $x \in \operatorname{spt}\mu$ and $0 < t \le \operatorname{diam}(\operatorname{spt}\mu)$ define
$$\beta_1(x,t) = \inf_L \frac{1}{t^n}\int_{B(x,t)} \frac{\operatorname{dist}(y,L)}{t}\, d\mu(y)$$
and
$$\beta_\infty(x,t) = \inf_L \sup_{y\in\operatorname{spt}\mu\cap B(x,t)} \frac{\operatorname{dist}(y,L)}{t},$$
where the infimum is taken over all the n-planes $L \subset \mathbb{R}^d$. Show that
$$\beta_\infty(x,t) \le C\beta_1(x,2t)^{1/(n+1)}.$$
(These are natural quantities which measure how "flat" the measure μ is on $B(x,t)$.)