## EXERCISES FOR MARCH 29

The first two exercises are about $\beta$-numbers.

**Exercise 0.1.** Show, by example, that the number $\delta > 0$ in Jones' $L^\infty$ traveling salesman theorem (Theorem 2.5 or Theorem 4.1 in the lecture notes) is necessary. In other words, the theorem fails for $\delta = 0$.

Recall that a Radon measure $\mu$ is *smooth*, if it is doubling, and there is a constant $\theta > 0$ such that $\mu(B(x, \theta r)) \leq \mu(B(x,r))/2$ for all $x \in \operatorname{spt} \mu$ and $0 < r \leq \operatorname{diam}(\operatorname{spt} \mu)$.

**Exercise 0.2.** This is Lemma 5.8 from the lecture notes; for hints, see the proof sketch there. Let $\mu$ be a smooth measure, and let $P, R$ be (non-dyadic) cubes with $\ell(P) \sim \ell(R) \lesssim \operatorname{diam}(\operatorname{spt} \mu)$, and let $\ell_P, \ell_R$ be lines, which minimise $\beta_{\mu,1}(P)$ and $\beta_{\mu,1}(R)$, respectively. Assume that

$$\tau P \cap \tau R$$

contains a point of $\operatorname{spt} \mu$ for some $\tau < 1$. Then, $\ell_P$ and $\ell_R$ are very close in the following sense:

$$\operatorname{dist}(z, \ell_R) \lesssim_\tau \min\{\beta_{\mu,1}(P)\ell(P), \beta_{\mu,1}(R)\ell(R)\}, \quad z \in \ell_P \cap P.$$

By symmetry, the same holds for $\operatorname{dist}(z, \ell_P)$, for $z \in \ell_R \cap R$.

As a background for the next three exercises, you should keep in mind Theorem 3.12 from the book of Tolsa, which Henri proved on the first part of the course: if $\Gamma$ is an AD regular curve with $\operatorname{diam}(\Gamma) < \infty$, then there exists an $M$-Lipschitz graph $G$ such that $\mathcal{H}^1(\Gamma \cap G) \geq \theta \operatorname{diam}(\Gamma)$, where $\theta, M > 0$ only depend on the AD-regularity constants of $\Gamma$. Here an "$M$-Lipschitz graph" is a set of the form

$$G = R(\{(x, f(x)) : x \in \mathbb{R}\}),$$

where $R$ is any rotation or translation, and $f \colon \mathbb{R} \to \mathbb{R}$ is $M$-Lipschitz.

Motivated by this, let's make the following definition:

**Definition 1.** Fix $\theta > 0$. A bounded set $E \subset \mathbb{R}^2$ is called *graph $(\theta, M)$-rectifiable*, if there exists an $M$-Lipschitz graph $G$ such that $\mathcal{H}^1(E \cap G) \geq \theta \operatorname{diam}(E)$.

Thus, *a bounded AD regular curve is $(\theta, M)$-rectifiable with $\theta, M$ depending only on the AD regularity constants*. The next exercises investigate, to what extent **large subsets** of AD regular curves are still graph $(\theta, M)$-rectifiable. I hope that your first instinct is the same as mine: they should also be graph $(\theta, M)$-rectifiable with slightly worse constants, right? Well, if you thought so, think again...

**Exercise 0.3.** Assume that $\Gamma$ is a Lipschitz curve (not even necessarily AD regular), and $E \subset \Gamma$ is a bounded subset with $\mathcal{H}^1(E) > 0$. Prove that $E$ is graph $(\theta, M)$ rectifiable for some $\theta > 0$ and $M < \infty$.[1]

[1] To make things a little easier, you may assume that $\Gamma$ is a $C^1$-curve. The general case reduces to this by a standard approximation result, but I don't want to make that a prerequisite for the exercise.

So, subsets of Lipschitz curves are graph $(\theta, M)$-rectifiable for some $(\theta, M)$, but these numbers turn out to be **impossible** to quantify! The next exercise is just a lemma for the last one.

**Exercise 0.4.** Let $E \subset \mathbb{R}^2$ be a graph $(\theta, M)$-rectifiable bounded set. Prove that there is a line $\ell \subset \mathbb{R}^2$ such that the orthogonal projection $\pi_\ell(E)$ on to $\ell$ has length

$$\mathcal{H}^1(\pi_\ell(E)) \gtrsim \left(\frac{\theta}{M}\right) \operatorname{diam}(E).$$

**Exercise 0.5.** Pick a large number $n$. Start the unit line segment $[0, 1] \subset \mathbb{R}^2$, split it into $n$ pieces of length $1/n$, and rotate each piece counterclockwise by an angle $2\pi/n$. Then, split these pieces again into $n$ pieces of length $1/n^2$, and rotate by $2\pi/n$. Repeat $n$ times (so, heuristically, the total amount of rotation equals $n \cdot (2\pi/n) = 2\pi$). This process gives you a set $E_n$. Prove that $\mathcal{H}^1(E_n) = 1$, and that $E_n \subset \Gamma_n$ for for some AD regular curve $\Gamma_n$ with $\operatorname{diam}(\Gamma_n) \sim 1$, where the regularity constants do not depend on $n$.

Finally, prove that $\mathcal{H}^1(\pi_\ell(E)) \lesssim 1/n$ for **every** line $\ell \subset \mathbb{R}^2$.

In particular, taking $n$ large, $E_n$ is not graph $(\theta, M)$ rectifiable for any "fixed" constants $\theta, M$, even if the curves $\Gamma_n$ are, by the result in Tolsa's book! It's fun to try to visualise, where the Lipschitz graph $G_n$ "lives", which satisfies $\mathcal{H}^1(\Gamma_n \cap G_n) \gtrsim 1$...