# GEOMETRIC MEASURE THEORY AND SINGULAR INTEGRALS: EXERCISE SET 1

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**Exercise 5.** Suppose $n$ is an integer and $\mu$ is a measure satisfying $cr^n \leq \mu(B(x,r)) \leq Cr^n$ for all $x \in \operatorname{spt}\mu$ and $0 < r \leq \operatorname{diam}(\operatorname{spt}\mu)$. For $x \in \operatorname{spt}\ \mu$ and $0 < t \leq \operatorname{diam}(\operatorname{spt}\mu)$ define

$$\beta_1(x,t) = \inf_L \frac{1}{t^n} \int_{B(x,t)} \frac{\operatorname{dist}(y,L)}{t}\, d\mu(y)$$

and

$$\beta_\infty(x,t) = \inf_L \sup_{y \in \operatorname{spt}\mu \cap B(x,t)} \frac{\operatorname{dist}(y,L)}{t},$$

where the infimum is taken over all the $n$-planes $L \subset \mathbb{R}^d$. Show that

$$\beta_\infty(x,t) \leq C\beta_1(x,2t)^{1/(n+1)}.$$

(These are natural quantities which measure how "flat" the measure $\mu$ is on $B(x,t)$.)

**Solution.** Let $x \in \operatorname{spt}\mu$ and $t \in (0, \operatorname{diam}(\operatorname{spt}\mu)]$. We first make the observation that $\beta_\infty(x,t) \leq 1$. Indeed, let $L$ be any $n$-plane through $x$. Then

$$\sup_{y \in \operatorname{spt}\mu \cap B(x,t)} \frac{d(y,L)}{t} \leq \sup_{y \in \operatorname{spt}\mu \cap B(x,t)} \frac{|y-x|}{t} \leq 1.$$

This, by the definition of $\beta_\infty(x,t)$, proves the observation.

Assume now $x \in \operatorname{spt}\mu$ and $t \in (0, \operatorname{diam}(\operatorname{spt}\mu)/2]$. Fix an $n$-plane $L$ so that

$$\frac{1}{(2t)^n} \int_{B(x,2t)} \frac{d(y,L)}{t}\, \mathrm{d}\mu(y) \leq 2\beta_1(x,2t). \tag{0.1}$$

With this $n$-plane $L$, choose a point $y_0 \in \operatorname{spt}\mu \cap B(x,t)$ so that

$$\frac{d(y_0,L)}{t} \geq 2^{-1} \sup_{y \in \operatorname{spt}\mu \cap B(x,t)} \frac{d(y,L)}{t}.$$

Suppose first $d(y_0,L) \geq 2t$. In this case if $y \in B(y_0,t)$, then $d(y,L) \geq t$. Using this, and the fact that $B(y_0,t) \subset B(x,2t)$, we have

$$\frac{1}{t^n} \int_{B(x,2t)} \frac{d(y,L)}{t}\, \mathrm{d}\mu(y) \geq \frac{1}{t^n} \int_{B(y_0,t)} \frac{d(y,L)}{t}\, \mathrm{d}\mu(y) \geq \frac{\mu(B(y_0,t))}{t^n} \sim 1.$$

By our choice of $L$ in (0.1) it follows that $\beta_1(x,2t) \gtrsim 1$. Combining this with the observation $\beta_\infty(x,t) \leq 1$ gives

$$\beta_\infty(x,t)^{n+1} \leq 1 \lesssim \beta_1(x,2t).$$

Suppose then $d(y_0, L) < 2t$. If $y \in B(y_0, d(y_0, L)/2)$, then $d(y, L) \sim d(y_0, L)$. Since $B(y_0, d(y_0, L)/2) \subset B(x, 2t)$, we can estimate

$$\begin{aligned}
\frac{1}{t^n} \int_{B(x,2t)} \frac{d(y, L)}{t} \, \mathrm{d}\mu(y) &\geq \frac{1}{t^n} \int_{B(y_0, d(y_0, L)/2)} \frac{d(y, L)}{t} \, \mathrm{d}\mu(y) \\
&\sim \frac{d(y_0, L)\mu(B(y_0, d(y_0, L)/2))}{t^{n+1}} \\
&\sim \Big(\frac{d(y_0, L)}{t}\Big)^{n+1}.
\end{aligned}$$

By the choice of $L$ and $y_0$, this concludes the proof.