UH/ Department of Mathematics and Statistics
Introduction to mathematical finance I, spring 2016
Exercise -2 (3.2.2016)

1. We shall use this modified version of Farkas lemma:

   Let $\widetilde{S}$ be a $(d+1)\times n$ matrix, and $\widetilde{\pi}=(\widetilde{\pi}_0,\widetilde{\pi}_1,\dots,\widetilde{\pi}_d)\in\mathbb{R}^{d+1}$.

   Either of these two alternatives always holds:

   (a) There exists a vector $q=(q_1,\dots,q_n)^\top>0$ (with $q_i>0\ \forall i$) such that $\widetilde{S}q=\widetilde{\pi}$

   (b) There exists a vector $\xi=(\xi_0,\xi_1,\dots,\xi_d)\in\mathbb{R}^{d+1}$ such that $\xi\widetilde{S}\in\mathbb{R}^n_+\setminus\{0\}$ and $\widetilde{\pi}\cdot\xi\le 0$.

   This can be proved by using the separating hyperplane theorem, for an open convex set not containing the origin.

   Consider now a finite probability space $\Omega=\{\omega_1,\omega_2,\dots,\omega_n\}$, with a reference probability measure $\mathbb{P}$ such that $\mathbb{P}(\{\omega_i\})>0\ \forall i=1,\dots,n$, which means that all events $\{\omega_i\}$ are possible.

   Consider a market model with $(d+1)$ instruments $S_t^{(0)}(\omega),S_t^{(0)}(\omega),\dots,S_t^{(0)}(\omega)$, where $t=0,1$ is a time parameter, and respective initial deterministic prices $S_0^{(0)}=\pi^{(0)},S_0^{(1)}=\pi^{(1)},\dots,S_0^{(d)}=\pi^{(d)}$.

   Assuming that $S_t^{(0)}(\omega)\neq 0\ \forall\omega\in\Omega$ (and of course also the price $\pi^{(0)}\neq 0$, and we choose it as *numeraire* to define the discounted prices for $k=0,1,\dots,d$ as

   $$\widetilde{S}_1^{(k)}(\omega)=\frac{S_1^{(k)}(\omega)}{S_1^{(0)}(\omega)},\quad \widetilde{\pi}^{(k)}=\frac{\pi^{(k)}}{\pi^{(0)}}=\widetilde{S}_0^{(k)}=\frac{S_0^{(k)}}{S_0^{(0)}},$$

   Note that for the numeraire instrument we have

   $$\widetilde{S}_t^{(0)}(\omega)=\widetilde{\pi}^{(0)}=1$$

   Use Farkas lemma to give an alternative proof of the first fundamental theorem of mathematical finance on a finite probability space, namely that

   the (discrete) market model above is arbitrage free if and only if there exists a probability Q with $q_i=Q(\{\omega_i\})>0\ \forall i$ such that

   $$\widetilde{\pi}^{(k)}=E_Q\big(\widetilde{S}_1^{(k)}\big),\quad \forall k=1,\dots,d.$$

2. We consider a finite probability space with $\Omega = \{\omega_1, \omega_2, \omega_3\}$ probability space with $P(\omega_i) = 1/3, \quad i = 1, 2, 3,$

   and two financial instruments $S_t(\omega), U_t(\omega)$, ($t = 0, 1$ is the time parameter), and at time $t = 1$

$$S_1(\omega_1) = 1, \quad S_1(\omega_2) = 3/2, \quad S_1(\omega_3) = 1/2$$
$$U_1(\omega_1) = 0, \quad U_1(\omega_2) = 1/2, \quad U_1(\omega_3) = 1.$$

   We haven't fixed yet the initial prices (S_0, U_0) at time $t = 0$. Also at this stage we are not allow to deposit or borrow money from the bank, we consider only portfolios where all the capital is invested in the S_t and U_t instruments.

   (a) Find the pairs of initial prices (S_0, U_0) at time $t = 0$ for the random pair $(S_1(\omega), U_1(\omega))$ such that the market model is arbitrage free.

   (b) Choose $S_t(\omega)$ as numeraire and find for each pair of arbitrage-free prices (S_0, U_0) the correponding set of risk-neutral measures Q.

   Hint: find first the support (the set of all possible values) of the probability distribution of the discounted stock price U_1/S_1.

   (c) For which of the arbitrage free initial prices the market (S_t, U_t) is complete (i.e. the risk neutral measure is unique) ?

   (d) Is it possible to choose $U_t(\omega)$ as numeraire ?

   (e) We now add one financial instument and consider the market model (S_t, U_t, B_t), where B_t is the riskless investment with $B_0 = 1$ and $B_1 = (1 + r)$, $r = 0.8$.

   Find the set of arbitrage-free initial prices (S_0, U_0) in the market model with three instruments (S_t, U_t, B_t)

   Hint: choose for example U_t as numeraire find first the support of the joint probability distribution of the discounted stock prices $\left(\frac{U_1}{S_1}, \frac{B_1}{S_1}\right)$, or alternatively, use B_t as numeraire and find first the support of the joint probability distribution of the discounted stock prices $\left(\frac{U_1}{B_1}, \frac{S_1}{B_1}\right)$, and then look at the interior of the convex hull of the support.

   (f) For each pair of arbitrage free initial prices (S_0, U_0) in the market with three instruments (S_t, U_t, B_t) find the set of corresponding risk-neutral measures with repect to the numeraire S_t, and also with respect to the numeraire B_t.

   (g) With $B_0 = 1$ and $B_1 = 1.8$ fixed, for which arbitrage-free initial prices (S_0, U_0) the market (S_t, U_t, B_t) is complete ?