UH/ Department of Mathematics and Statistics
Introduction to mathematical finance I, spring 2016
Exercise -1 (28.1.2016)
Notation: $\mathbb{R}_+ = [0, \infty)$.

1. Let V be a vector space, for example $V = \mathbb{R}^d$. A set $\mathcal{C} \subseteq V$ is convex if and only if
$$x, y \in \mathcal{C},\ 0 \le \alpha \le 1 \Longrightarrow \alpha x + (1-\alpha) y \in \mathcal{C}$$
Show that for $n \in \mathbb{N}$,
$$x_i \in \mathcal{C},\ \alpha_i \ge 0, i = 1, \dots, n \text{ and } \sum_{i=1}^n \alpha_i = 1,$$
$$\Longrightarrow \sum_{i=1}^n \alpha_i x_i \in \mathcal{C}$$

2. *Farkas' lemma*

   Let A be a $(d \times n)$ matrix, and $b = (b_1, \dots, b_d) \in \mathbb{R}^d$.

   Either of these two alternatives always holds:

   (a) There is $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n_+$ such that $j = 1, \dots, n$ jolla $Ax = b$

   (b) There is $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ such that $yA \in \mathbb{R}^d_+$ and $b \cdot y < 0$.

   Prove Farkas' lemma by using the separating hyperplane theorem.

   Hint Think about the geometry of the problem: if $a_1, \dots, a_n \in \mathbb{R}^d$ are the column vectors of the matrix A, you can show that
$$\mathcal{C} = \left\{ \sum_{i=1}^n \alpha_i a_i :\ \alpha_i \in \mathbb{R}_+ \right\} \subseteq \mathbb{R}^d$$
   which is the convex cone generated by the vectors $a_1, \dots, a_n$, is actually convex and closed in $\mathbb{R}^d$.

   and the alternatives (a) and (b) correspond to the cases where $b \in \mathcal{C}$ and $b \notin \mathcal{C}$, respectively.

3. Prove Gordon theorem: for a matrix $A \in \mathbb{R}^{d \times n}$,

   either $Ax > 0$ for some $x \in \mathbb{R}^n$, ($r = (r_1, \dots, r_d) > 0$ means $r_i > 0\ \forall i$),

   or $yA = 0$ for some $y \in \mathbb{R}^d_+ \setminus \{\ 0\}$.

website	a	b	c	d	e	f	g
Barcelona wins	1.85	1.80	1.95	1.80	1.85	1.85	1.75
Manchester City wins	4.30	4.55	4.35	4.30	4.55	4.60	4.70
Draw	3.50	3.55	3.35	3.70	3.30	3.45	3.55

Table 1: gambling multipliers

4. A betting-website offers the following multiplier coefficients for the football game Barcelona-Manchester City: 1.85 for a Barcelona win, 4.3 for a Manchester city win, 3.5 for a draw, Is this pricing system arbitrage free ? Is it possible for a gambler to construct an arbitrage strategy with non-negative bets (without short positions?)

5. Table (1) shows the coefficients for Barcelona-Manchester-City game offered by 7 different gambling websites:

Check whether a gambler can find an arbitrage possibility with non-negative bets (without short positions) by using the highest multipliers offered for each result.