UH Introduction to mathematical finance I, Exercise-5 (24.02.2016)

In all the exercises we consider random variables defined on a probability space $(\Omega, \mathcal{F})$ equipped with a probability measure $\mathbb{P}$ and a filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$, where $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$.

Recall that a stochastic process $(M_t : t \in \mathbb{N})$ is a $(P, \mathbb{F})$-martingale if $M_t \in L^1(\Omega, \mathcal{F}_t, P)$ $\forall t \in \mathbb{N}$ and $E_P(M_t|\mathcal{F}_{t-1}) = M_{t-1}$ $\forall t \geq 1$.

1. Let $W_1 \sim \mathcal{N}(0,1)$ be a standard Gaussian random variable with $E_P(W_1) = 0$ and $E_P(W^2) = 1$. Recall that $E_P(\exp(\theta W_1)) = \exp(\theta^2/2)$. Consider a market model $(S_t, B_t : t \in \{\,0, 1\})$ where $B_0 = S_0 = 1$, $B_t = B_0(1+r)$, $r > -1$ is deterministic.

   and
   $$S_1 = S_0 \exp(\sigma W_1 + \mu - \frac{\sigma^2}{2}).$$
   Determine a risk neutral measure $Q \sim P$ sucj that W_1 is Gaussian also under Q.

   Hint : try a measure Q^θ with likelihood ratio (Radon-Nikodym derivative) $\frac{dQ^\theta}{dP} = \zeta_1(\theta) = \exp(\theta W_1)$, and show that with respect to Q^θ W_1 is also Gaussian, and compute for wich θ value Q^θ is risk-neutral.

2. Compute the set of arbitrage free prices for the european call and put options $(S_1 - K)^+$ ja $(K - S_1)^+$, and compute the cheapest supehdging strategy and the most expansive subhedging strategy.

3. On a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $F = (\mathcal{F}_t : t \in \mathbb{N})$, $\Delta W_t(\omega)$ $t = 1, \dots, T$ standard Gaussian random variables and let $W_t = W_1 + W_2 + \cdots + W_t$. Under P S_t is Gaussian with $E_P(S_t) = 0$ and variance $E_P(S_t^2) = t$. We assume that W_t is $\mathcal{F}_t$-measurable and ΔW_t is P-independent from the σ-algebra $\mathcal{F}_{t-1}$. Let $(S_t, B_t : t \in \{\,0, 1\})$ be a market model where $B_0 = S_0 = 1$, $B_t = B_{t-1}(1 + r_t)$, $r_t > -1$ is deterministic,

   and $S_t = S_0 \exp\big(\sum_{u=1}^t \sigma_u \Delta W_u + \sum_{u=1}^t (\mu_u - \frac{\sigma_u^2}{2})\big)$

   (a) Construct a risk-neutral measure Q under which ΔW_t are Gaussian with ΔW_t is Q-independent from the σ-algebrasta $\mathcal{F}_{t-1}$.

   Hint Construct a likelihood process Z_t with product form, where $Z_0 = 1$ and
   $$Z_t = Z_1 \frac{Z_2}{Z_1} \frac{Z_2}{Z_1} \frac{Z_t}{Z_{t-1}} = \zeta_1 \zeta_2 \times \cdots \times \zeta_t,$$

such that $Z_t(\omega) \geq 0$, $E_P(Z_t) = 1$ and $E_Q(S_T|\mathcal{F}_{t-1}) = S_t \frac{B_t}{B_T}$. Use Bayes formula

$$E_Q(S_t|\mathcal{F}_{t-1}) = E_Q(S_t|\mathcal{F}_{t-1}) = \frac{E_P(S_t Z_t|\mathcal{F}_{t-1})}{E_P(Z_t|\mathcal{F}_{t-1})}$$

(b) What happens if μ_t,σ_t r_t are $\mathbb{F}$-predictable but not deterministic, , is Q riskineutral also in this more general case?

(c) Assuming that $\forall t, \mu_t = \mu, \sigma_t = \sigma$ $r_t = r$ are determinic constants, for $t < T$, use the riskneutral measure Q as a pricing measure and compute the corresponding arbitrage-free prices $c_{\text{call}} \frac{B_t}{B_T} E_Q((S_T - K)^+|\mathcal{F}_t)$ ja $c_{\text{put}} E_Q((K - S_T)^+|\mathcal{F}_t)$ for the european call- and put-options $(S_T - K)^+$ ja $(K - S_T)^+$ (Black and Scholes formulae). This market is incomplete, and these european options are not replicable, the arbitrage free prices are not unique, since the risk-neutral martingale measure is not unique.

4. Let $Y_1, \dots, Y_T$ binary random variables with $P(Y_t = 1|\mathcal{F}_{t-1}) = 1 - P(Y_t = 0|\mathcal{F}_{t-1}) = p_t(\omega) \in (0, 1)$. We assume that Y_t on $\mathcal{F}$-measurable and $p_t(\omega)$ is $\mathcal{F}_{t-1}$-measurable, $\forall t = 1, \dots, T$.

   In the market model $(B_t, S_t, X_t : t = 0, 1, \dots, T)$ the dynamics of these financial instruments is the following: $B_0 = S_0 = X_0 = 1$. and

   $$B_t = B_{t-1}(1 + r_t), S_t = S_{t-1}(1 + u_t)^{Y_t}(1 + d_t)^{1-Y_t}, X_t = X_{t-1}(1 + d_t)^{Y_t}(1 + u_t)^{1-Y_t},$$

   where $-1 < d_t(\omega) < r_t(\omega) < u_t(\omega)$ are $\mathbb{F}$-predictable processes.

   (a) Compute a risk-neutral martingale measure for this market.

   Hint Construct a likelihood process of product form.

   (b) Assuming that $u_t = u, d_t = d, r_t = r, p_t = p$ where $-1 < d_t < r_t < u_t$ and $p \in (0, 1)$ are deterministic constants, for $t \leq T$ compute the arbirtage free price $c(\text{swap}) = \frac{B_t}{B_T} E_P((S_T - X_T)^+|\mathcal{F}_t)$ of the swap-option $(S_T - X_T)^+$.

5. Let $(X_t : t \in \mathbb{N})$ independent and identically distributed random variables with $P(X_t = 1) = 1 - P(X_t = -1) = p = 1/2$, and $S_t = X_1 + X_2 + \cdots + X_t$. For $a < 0 < b$, where $a, b \in \mathbb{Z}$, consider the random time

   $$\tau(\omega) = \inf\{t \in \mathbb{N} : S_t(\omega) \notin (a, b)\}.$$

(a) Show that $\tau(\omega)$ is a stopping time in the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$ jossa $\mathcal{F}_t = \sigma(S_u : u \leq t) = \sigma(X_u : u \leq t)$.

(b) Show that S_t is a $\mathbb{F}$-martingale and it is square integrable $E(S_t^2) < \infty \ \forall t$.

(c) Show that the stopped process $(S_{t\wedge\tau} : t \in \mathbb{N})$ is a martingale.

(d) Show that $P(\tau < \infty) = 1$. Hint: you can use the second Borel Cantelli lemma.

(e) Compute $P(S_\tau = a)$ and $P(S_\tau = b)$. Hint: show that $S_{t\wedge\tau}$

(f) Show that the martingale S_t has $\mathbb{F}$-predictable variation $\langle S \rangle_t = t$ which by definition means that

$$M_t := S_t^2 - t$$

is a $\mathbb{F}$-martingale.

(g) Show that $E(\tau) < \infty$. hint: $(M_{t\wedge\tau} : t \in \mathbb{N})$ is a martingale, and we have the upper and lower bounds

$$0 \leq n \wedge \tau = S_{n\wedge\tau}^2 - M_{n\wedge\tau}, \text{ where } S_t^2 \leq \max\{a^2, b^2\} \forall t \qquad (0.1)$$

use Fatou lemma for $n \to \infty$.

(h) Compute the expectation $E(\tau)$. Hint compute $E(S_\tau^2)$, and take the expectation in (0.1), and use monotone convergence theorem and Lebesgue dominated convergence theorem.