UH Introduction to mathematical finance I, Exercise-4 (17.02.2016)

In all the exercises we consider random variables defined on a probability space $(\Omega, \mathcal{F})$ equipped with a probability measure $\mathbb{P}$ and a filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$, where $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \leq t$.

Recall that a stochastic process $(M_t : t \in \mathbb{N})$ is a $(P, \mathbb{F})$-martingale if $M_t \in L^1(\Omega, \mathcal{F}_t, P)$ $\forall t \in \mathbb{N}$ and $E_P(M_t|\mathcal{F}_{t-1}) = M_{t-1}$ $\forall t \geq 1$.

1. Prove: if M_t is a $(P, \mathbb{F})$-martingale, then $E_P(M_t) = E_P(M_0)$ $\forall t \in \mathbb{N}$.

   Solution: Obvious by iteration of the martingale property.

2. Prove the following lemma

   If $(M_t(\omega) : t \in \mathbb{N})$ is a $(\mathbb{F}, P)$-martingale which is also $\{\mathcal{F}_t\}$-predictable, meaning that $\forall t > 0$ $M_t(\omega)$ on $\mathcal{F}_{t-1}$-measureable, then it must be a random constant: $M_t(\omega) = M_0(\omega)$ $\quad \forall t \in \mathbb{N}$, where $M_0(\omega)$ is $\mathcal{F}_0$-measurable.

   Solution: Since $M_t(\omega)$ is $\mathcal{F}_{t-1}$-measurable, then $M_t = E_P(M_t|\mathcal{F}_{t-1}) = M_{t-1}$. The claim follows by iteration.

3. Let $\mathbb{G} = (\mathcal{G}_t : t \in \mathbb{N})$ be a smaller filtration, such that $\mathcal{G}_t \subseteq \mathcal{F}_t$ $\forall t \in \mathbb{N}$. Show that if M_t if $(P, \mathbb{F})$-martingale which is $\mathbb{G}$-adapted, is also a $(P, \mathbb{G})$-martingale.

   Solution: If M_t is a $(P, \mathbb{F})$-martingale, then $M_t \in L^1(\Omega, P, \mathbb{G})$. By the tower property we have

   $$E_P(M_t|\mathcal{G}_{t-1}) = E_P(E_P(M_t|\mathcal{F}_{t-1})|\mathcal{G}_{t-1}) = E_P(M_{t-1}|\mathcal{F}_{t-1}) = M_{t-1}.$$

4. Prove the following lemma: Let $\mathcal{G} \subset \mathcal{F}$ a sub-σ-algebra, $Y(\omega)$ a $\mathcal{G}$-measurable random variable and let $X(\omega)$ be P-independent from the σ-algebra $\mathcal{G}$. For example it could be that $\mathcal{G} = \sigma(Y)$ with $X \overset{P}{\perp\!\!\!\perp} Y$.

   For all bounded Borel-measurable functions $f(x, y)$ we have

   $$\begin{aligned} E_P(f(X, Y)|\sigma(Y))(\omega) &= \int_\Omega f(X(\widetilde{\omega}), Y(\omega))P(d\widetilde{\omega}) \\ &= \int_\Omega f(X(\widetilde{\omega}), y)P(d\widetilde{\omega})\bigg|_{y=Y(\omega)} = \int_{\mathbb{R}^d} f(x, Y(\omega))P_X(dx) \end{aligned}$$

   where $P_X(B) = P(\omega : X(\omega) \in B)$ for every Borel set $B \subseteq \mathbb{R}^d$, meaning that we fix the value $Y(\omega) = y$ and integrate the random variable X from the marginal distribution.

   Hint: Apply the definition of conditional expectation.

   You can start assuming that f has the product form $0 \leq f(x, y) = g(x)h(x)$, with g, h measurable functions. Then we know that jointly measurable functions can be approximated from below by sums of product functions

   $$f_n(x, y) = \sum_{k=1}^{n} g_n(x)h_n(x) \uparrow f(x, y) \quad \forall x, y$$

and use the monotone convergence theorem.

Solution: Given $f(x, y)$, we decompose it as $f(x, y) = f^+(x, y) - f^-(x, y)$ where $f^+(x, y) = \max(f(x, y), 0)$ and $f^-(x, y) = \max(-f(x, y), 0)$, so that both the functions $f^+(x, y)$ and $f^-(x, y)$ are bounded and non-negative. This means that without loss of generality we can assume $f(x, y)$ to be bounded and non-negative. Furthermore, following the hint, for the momente we also assume that $f(x, y) = g(x)h(y)$. Then we have for any $B \in \sigma(Y)$

$$E_P(f(X, Y)\mathbf{1}_B(Y)) = E_P(g(X)h(Y)\mathbf{1}_B(Y)) = E_P(g(X))E_P(h(Y)\mathbf{1}_B(Y))$$

which means that

$$E_P(g(X)h(Y)|\sigma(Y)) = h(Y(\omega))E_P(g(X)) = h(Y(\omega)) \int g(x)P_X(dx).$$

For general $f(x, y)$ in order to get the claim, it's enough to apply the monotone converge theorem as suggested in the hint.

5. Consider an $\mathbb{F}$-adapted stochastic process $(X_t)_{t\geq 0}$ such that $X_t \in L^1(P)$ for all $t \geq 0$, $\Delta X_t = X_t - X_{t-1}$, and

$$A_t := \sum_{s=1}^{t} E_P(\Delta X_s|\mathcal{F}_{s-1}), \quad A_0 = 0$$

   (a) show that $A_n \in L^1$ and it is $\{\mathcal{F}_t\}$-predictable.
   Solution: $A_t \in L^1(P)$ because $X_t \in L^1(P)$ and A_t is $\mathcal{F}_t$-predictable since $E_P(\Delta X_s|\mathcal{F}_{s-1})$ are $\mathcal{F}_t$-predictable for $s = 1, \dots, t$.

   (b) show that $M_n := (X_n - X_0 - A_n)$ on $(P, \{\mathcal{F}_n\})$-martingale with $M_0 = 0$.
   Solution: M_n is in $L^1(P)$. Moreover,

$$\begin{aligned} E_P(M_n|\mathcal{F}_{n-1}) &= E_P(X_n - X_0|\mathcal{F}_{n-1}) - A_{n-1} - E_P(\Delta X_n|\mathcal{F}_{n-1}) \\ &= E_P(X_n - X_0|\mathcal{F}_{n-1}) - A_{n-1} - E_P(X_n - X_{n-1}|\mathcal{F}_{n-1}) \\ &= X_{n-1} - X_0 - A_{n-1} = M_{n-1}. \end{aligned}$$

   (c) The equation

$$X_n = X_0 + A_n + M_n$$

   is the Doob martingale decomposition of (X_t) into martingale and predictable part. Prove that the Doob decomposition is unique: if $(M')_n$ is another $(P, \{\mathcal{F}_n\})$-martingali and (A'_n) is another $\mathbb{F}$-predictable process such that $M'_0 = A'_0 = 0$ and

$$X_n = X_0 + A'_n + M'_n$$

   it follows that $M = M'$ and $A = A'$.
   Solution: We have necessarily that $A_n + M_n = A'_n + M'_n$ for $n \in \mathbb{N}_+$, then $A_n - A'_n = M'_n - M_n$, which implies that $M'{}_n - M_n$ is predictable. Since $M'_n - M_n$ a martingale, from exercise 2 we know that $M'_n - M_n = M'_0 - M_0 = 0$, then $M'_n = M_n$ and $A'_n = A_n$ for $n \in \mathbb{N}_+$.

(d) Show that when (X_n) is a submartingale (supermartingale, respectively) the predictable part A_n in the Doob decomposition A_n is non-decreasing (non-increasing respectively).

Solution: When X_n is a supermartingale we have $E_P(X_n|\mathcal{F}_{n-1}) \leq X_{n-1}$, so that

$$A_n = E_P(A_n|\mathcal{F}_{n-1}) = E_P(X_n - X_0 - M_n|\mathcal{F}_{n-1}) \leq X_{n-1} - X_0 - M_{n-1} = A_{n-1}$$

so A_n is non-increasing. Reversing the inequalities, if X_n is a submartingale, then A_n is non-decreasing.

6. Let X_0 and $(U_t : t \in \mathbb{N})$ $\mathbb{P}$-independent ranbdom variables with U_t uniformly distributed on $[0,1]$. Let $f_t : \mathbb{R} \times [0,1] \to \mathbb{R}$ be Borel measurable functions.

We define by induction $X_t(\omega) = f_t(X_{t-1}(\omega), U_t(\omega))$ $\forall t \geq 1$.

Let $\mathcal{F}_t = \sigma(X_s : 0 \leq s \leq t)$, and $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$.

(a) Show that $X_t(\omega)$ is a Markov process, which means

$$P\big(X_t \in B \big| \mathcal{F}_{t-1}\big)(\omega) = P\big(X_t \in B \big| \sigma(X_{t-1})\big)(\omega)$$

for all Borel sets B.

Solution: Since U_t is independent from $X_{t-1}, \ldots, X_0$, then we have

$$E_P(\mathbf{1}_B(X_t)) = E_P(\mathbf{1}_B(f_t(X_{t-1}, U_t)) = \int_0^1 du\, \mathbf{1}_B(f_t(X_{t-1}, u)) \in \sigma(X_{t-1}).$$

(b) Let $g : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Compute the Doob decomposition

$$g(X_t) = g(X_0) + A_t(g) + M_t(g)$$

where $A_t(g)$ is $\mathbb{F}$-predictable and $M_t(g)$ is a $\mathbb{F}$-martingale.

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$$g(X_t) = E_P\big(g(X_t)|\mathcal{F}_{t-1}\big) + \Big(g(X_t) - E_P\big(g(X_t)|\mathcal{F}_{t-1}\big)\Big)$$

where $g(X_t) = g\big(f(X_{t-1}, U_t)\big)$

Solution: We can construct A_t by using the standard formula

$$\begin{aligned} A_t &= \sum_{s=1}^t E_P(g(X_s) - g(X_{s-1})|\mathcal{F}_{s-1}) \\ &= \sum_{s=1}^t E_P(g(f(X_{s-1}, U_s)) - g(X_{s-1})|\mathcal{F}_{s-1}) \\ &= \sum_{s=1}^t E_P(g(f(X_{s-1}, U_s)) - g(X_{s-1})|\sigma(X_{s-1})) \\ &= \sum_{s=1}^t \int_0^1 du\, [g(f(X_{s-1}, u)) - g(X_{s-1})], \end{aligned}$$

then $M_t = g(X_t) - g(X_0) - \sum_{s=1}^t \int_0^1 du\, [g(f(X_{s-1}, u)) - g(X_{s-1})]$.

7. Let $Y_1(\omega), \ldots, Y_T(\omega)$ be P-independent and identically distributed binary random variables with $P(Y_t = 1) = 1 - P(Y_t = 0) = p \in (0, 1)$.

Consider the canonical probability space $\Omega = \{0,1\}^T$ of T-repeated coin tosses with $\omega = (\omega_1, \ldots, \omega_T)$, $\omega_t \in \{0, 1\}$ with the random variables defined as $Y_t(\omega) = \omega_t \in \{0, 1\}$.

Let $X_t(\omega) = Y_1 + Y_2 + \cdots + Y_t$.

(a) Show that X_t has Binomial(p, t) distribution meaning that

$$P(X_t = x) = \binom{t}{k} p^x (1-p)^{t-x}, \qquad \text{when } x \in \{0, 1, 2, \ldots, n\},\ P(X = x) = 0 \text{ otherwise}$$

Solution: The probability that the first x tosses give 1 and the remaining $t-x$ tosses give 0 is $p^x(1-p)^{t-x}$ and in this case $X_t = x$. To compute $P(X_t = x)$ we need to count all the possible comfigurations of x "objects"in t "places", which are $\binom{t}{x}$, so that

$$P(X_t = x) = \binom{t}{x} p^x (1-p)^{t-x} \quad \text{when } x \in \{0, 1, 2, \ldots, n\}, \quad P(X = x) = 0 \text{ otherwise}$$

(b) Compute the Doob martingale decomposition

$$X_t = X_0 + M_t + A_t$$

for the stochastic process $X_t(\omega)$, where(M_t) is a (P, F)-martingale and (A_t) is predictable with respect to the filtration $\mathbb{F} = (\mathcal{F}_t : t = 1, \ldots, T)$ with $\mathcal{F}_t = \sigma(Y_1, \ldots, Y_r)$.

Solution: We can construct the predictble part as follows

$$A_t = \sum_{s=1}^{t} \mathbb{E}_P(\Delta X_s | \mathcal{F}_{s-1}) = \sum_{s=1}^{t} \mathbb{E}_P(Y_s | \mathcal{F}_{s-1}) = \sum_{s=1}^{t} E_P(Y_s) = tp$$

then

$$M_t = X_t - X_0 - A_t.$$

(c) Compute the $\mathbb{F}$-predictable covariation process $\langle M \rangle_t$ such that

$$M_t^2 - \langle M \rangle_t$$

is a $(P, \mathbb{F})$-martingale.

Hint Compute the Doob decomposition of the process M_t^2.

Solution: Given the Doob decomposition of $M_t^2 = N_t + B_t$ where N_t is a $(P, \mathbb{F})$-martingale and B_t is predictable, then necessarily it results that

$$B_t = \langle M_t \rangle.$$

We observe that

$$\begin{aligned} M_t &= M_{t-1} - (M_{t-1} - M_t) = M_{t-1} + X_t - X_{t-1} + A_{t-1} - A_t \\ &= M_{t-1} + Y_t - p \end{aligned}$$

Then we have

$$\begin{aligned}\langle M_t \rangle = B_t &= \sum_{s=1}^{t} E_P(M_s^2 - M_{s-1}^2 | \mathcal{F}_{s-1}) = \sum_{s=1}^{t} E_P(2(Y_s - p)M_{s-1} + (Y_s - p)^2 | \mathcal{F}_{s-1}) \\ &= \sum_{s=1}^{t} 2M_{s-1}E_P(Y_s - p) + E_P(Y_s - p)^2 = tp(1-p)\end{aligned}$$