**UH Introduction to mathematical finance I, Exercise-5 (24.02.2016)**

In all the exercises we consider random variables defined on a probability space $(\Omega, \mathcal{F})$ equipped with a probability measure $\mathbb{P}$ and a filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$, where $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $s \le t$.

Recall that a stochastic process $(M_t : t \in \mathbb{N})$ is a $(P, \mathbb{F})$-martingale if $M_t \in L^1(\Omega, \mathcal{F}_t, P)$ $\forall t \in \mathbb{N}$ and $E_P(M_t|\mathcal{F}_{t-1}) = M_{t-1}$ $\forall t \ge 1$.

1. Let $W_1 \sim \mathcal{N}(0,1)$ be a standard Gaussian random variable with $E_P(W_1) = 0$ and $E_P(W^2) = 1$. Recall that $E_P(\exp(\theta W_1)) = \exp(\theta^2/2)$. Consider a market model $(S_t, B_t : t \in \{0,1\})$ where $B_0 = S_0 = 1$, $B_t = B_0(1+r)$, $r > -1$ is deterministic.

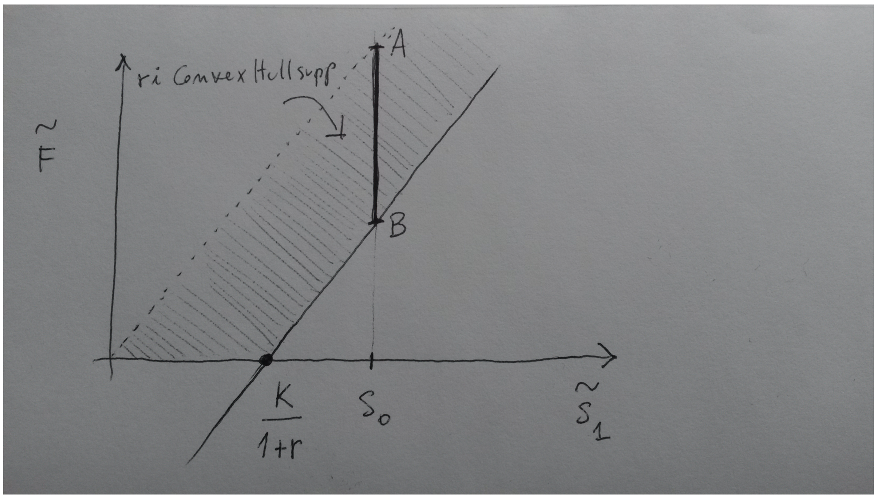
   and
   $$S_1 = S_0 \exp(\sigma W_1 + \mu - \frac{\sigma^2}{2}).$$
   Determine a risk neutral measure $Q \sim P$ such that $W_1$ is Gaussian also under $Q$.

   Hint : try a measure $Q^\theta$ with likelihood ratio (Radon-Nikodym derivative) $\frac{dQ^\theta}{dP} = \zeta_1(\theta) = \exp(\theta W_1 - \theta^2/2)$, and show that with respect to $Q^\theta$ $W_1$ is also Gaussian, and compute for wich $\theta$ value $Q^\theta$ is risk-neutral.

   **Solution:** Let us check that $W_1$ is Gaussian under $Q$:
   $$\mathbb{E}_Q(W_1) = \mathbb{E}_P(e^{\theta W_1 - \theta^2/2} W_1) = \frac{1}{2\pi} \int_{\mathbb{R}} dx\, e^{\theta x - \theta^2/2} x = \theta = \mu_Q$$
   and
   $$\mathbb{E}_Q(W_1^2) = \mathbb{E}_P(e^{\theta W_1 - \theta^2/2} W_1^2) = \frac{1}{2\pi} \int_{\mathbb{R}} dx\, e^{\theta x - \theta^2/2} x^2 = \theta^2 + 1,$$
   so that $\sigma_Q^2 = 1$. To check the Gaussianity we look at
   $$\mathbb{E}_Q(e^{\lambda W_1}) = \mathbb{E}_P(e^{(\theta+\lambda)W_1 - \theta^2/2}) = e^{\theta\lambda + \lambda^2/2} = e^{\lambda\mu_Q + \sigma_Q^2 \lambda^2/2}.$$
   To find a risk neutral measure we need to impose that
   $$1 = S_0 = \mathbb{E}_Q\left(\frac{S_1}{1+r}\right) = \frac{1}{2\pi(1+r)} \int_{\mathbb{R}} dx\, e^{(\sigma+\theta)x + \mu - (\sigma^2+\theta^2)/2 - x^2/2} = \frac{e^{\mu+\sigma\theta}}{1+r}$$
   so that $\theta$ turns out to be
   $$\theta = \frac{\ln(1+r) - \mu}{\sigma}.$$

2. Compute the set of arbitrage free prices for the european call and put options $(S_1 - K)^+$ ja $(K - S_1)^+$, and compute the cheapest superhedging strategy and the most expansive subhedging strategy.

   **Solution:** We consider first the option call: from the general theory we know that the arbitrage-free discounted price should lies in the relative interior of the convex hull of the support of the distribution:
   $$c(\widetilde{F}, S_0) \in \mathrm{ri}(\mathrm{ConvexHull}(\mathrm{supp}(\widetilde{F}, \widetilde{S}_1)))$$




Kuva 1: The segment $AB$ is the interval of arbitrage-free prices

where $\widetilde{F} = (S_1 - K)^+/(1+r)$ and $c(\widetilde{F})$ is the unknown price of the option. Since $S_1$ has a lognormal distribution, its support is $\mathbb{R}_+$ and then we can grafically represent the situation as follows (the picture depends on the value of $K/(1+r)$, in this example we set $K/(1+r) < S_0$):

From the picture is clear that the superhedging strategy would be $c(\widetilde{F}) = \widetilde{S}_1$ and the subhedging strategy $c(\widetilde{F}) = (\widetilde{S}_1 - K)^+/(1+r)$.

For the option put, we can exploit the parity relation

$$S_1 - K = (S_1 - K)^+ - (K - S_1)^+$$

so that the initial prices are such that

$$S_0 - K = c(F^{call}) - c(F^{put}).$$

Using the results for $c(F^{call})$, we can easily answer the analoguos questions for $c(F^{put}) = c(F^{call}) + K - S_0$.

3. On a probability space $(\Omega, \mathcal{F}, P)$ equipped with a filtration $F = (\mathcal{F}_t : t \in \mathbb{N})$, $\Delta W_t(\omega)$ $t = 1, \dots, T$ standard Gaussian random variables and let $W_t = W_1 + W_2 + \dots + W_t$. Under $P$ $S_t$ is Gaussian with $E_P(S_t) = 0$ and variance $E_P(S_t^2) = t$. We assume that $W_t$ is $\mathcal{F}_t$-measurable and $\Delta W_t$ is $P$-independent from the $\sigma$-algebra $\mathcal{F}_{t-1}$. Let $(S_t, B_t : t \in \{0, 1\})$ be a market model where $B_0 = S_0 = 1$, $B_t = B_{t-1}(1 + r_t)$, $r_t > -1$ is deterministic, and $S_t = S_0 \exp\left(\sum_{u=1}^t \sigma_u \Delta W_u + \sum_{u=1}^t (\mu_u - \frac{\sigma_u^2}{2})\right)$

   (a) Construct a risk-neutral measure $Q$ under which $\Delta W_t$ are Gaussian with $\Delta W_t$ is $Q$-independent from the $\sigma$-algebrasta $\mathcal{F}_{t-1}$.

   **Hint** Construct a likelihood process $Z_t$ with product form, where $Z_0 = 1$ and

$$Z_t = Z_1 \frac{Z_2}{Z_1} \frac{Z_2}{Z_1} \frac{Z_t}{Z_{t-1}} = \zeta_1 \zeta_2 \times \dots \times \zeta_t,$$

such that $Z_t(\omega) \geq 0$, $E_P(Z_t) = 1$ and $E_Q(S_T|\mathcal{F}_{t-1}) = S_t \frac{B_t}{B_T}$. Use Bayes formula

$$E_Q(S_t|\mathcal{F}_{t-1}) = E_Q(S_t|\mathcal{F}_{t-1}) = \frac{E_P(S_t Z_t|\mathcal{F}_{t-1})}{E_P(Z_t|\mathcal{F}_{t-1})}$$

**Solution:** We want the measure $Q$ to be such that

$$(1 + r_t)S_{t-1} = E_Q(S_t|\mathcal{F}_{t-1}) = E_P(S_t Z_t|\mathcal{F}_{t-1}) = \frac{E_P(S_t Z_t|\mathcal{F}_{t-1})}{E_P(Z_t|\mathcal{F}_{t-1})} \tag{0.1}$$

where we used the hint. From the lecture, we know that $Z_t$ is a martingale, so we have

$$(1 + r_t)S_{t-1} = E_P(S_t \frac{Z_t}{Z_{t-1}}|\mathcal{F}_{t-1}) = E_P(S_t \zeta_t|\mathcal{F}_{t-1}).$$

As we have done for exercise 1, we see that the $\zeta_t$ we are after is $\zeta_t = e^{\theta_t \Delta W_t - \theta_t^2/2}$ where $\theta_t = \sigma_t^{-1}(\ln(1 + r_t) - \mu_t)$.

(b) What happens if $\mu_t$,$\sigma_t$ $r_t$ are $\mathbb{F}$-predictable but not deterministic, , is $Q$ riskineutral also in this more general case?

**Solution:** It is risk neutral because they just come out of the conditional expectation in

(c) Assuming that $\forall t, \mu_t = \mu$,$\sigma_t = \sigma$ $r_t = r$ are determinic constants, for $t < T$, use the riskneutral measure $Q$ as a pricing measure and compute the corresponding arbitrage-free prices $c_{\text{call}} = \frac{B_t}{B_T} E_Q((S_T - K)^+|\mathcal{F}_t)$ and $c_{\text{put}} = E_Q((K - S_T)^+|\mathcal{F}_t)$ for the european call- and put- options $(S_T - K)^+$ ja $(K - S_T)^+$ (Black and Scholes formulae). This market is incomplete, and these european options are not replicable, the arbitrage free prices are not unique, since the risk-neutral martingale measure is not unique.

**Solution:** Note that we can write

$$S_T = S_t \exp((\mu - \frac{\sigma^2}{2})\tau + \sigma(W_T - W_t)) = S_t \exp((\mu - \frac{\sigma^2}{2})\tau + \sigma W_\tau)$$

where $\tau = T - t$.

By using the Bayes formula as before and being $Z_\tau = e^{\theta W_\tau - \tau\theta^2/2}$ with $\theta = \sigma^{-1}(\ln(1 + r) - \mu)$, we get

$$\begin{aligned}
\mathbb{E}_Q(S_T|\mathcal{F}_t) &= \mathbb{E}_P(S_T Z_\tau|\mathcal{F}_t) = \mathbb{E}_P(Z_\tau S_t \exp((\mu - \frac{\sigma^2}{2})\tau + \sigma W_\tau)|\mathcal{F}_t) \\
&= S_t \mathbb{E}_P(\exp((\mu - (\sigma^2 + \theta^2)/2)\tau + (\sigma + \theta)W_\tau)|\mathcal{F}_t) \\
&= S_t(1 + r)^\tau
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}_Q(S_T^2|\mathcal{F}_t) &= \mathbb{E}_P(S_T^2 Z_\tau|\mathcal{F}_t) \\
&= S_t^2 \mathbb{E}_P(\exp((2\mu - \sigma^2 - \theta^2/2)\tau + (2\sigma + \theta)W_\tau)|\mathcal{F}_t) \\
&= S_t^2(1 + r)^{2\tau} e^{\sigma^2\tau}
\end{aligned}$$

thus, under $Q$, at $t$ the price of the stock at expiry $S_T$ follows a lognormal distribution with mean

$$S_t(1+r)^\tau = e^{\ln S_t + \tau \ln(1+r)} \tag{0.2}$$

and variance

$$S_t^2(1+r)^{2\tau}(e^{\sigma^2\tau}-1) = (e^{\sigma^2\tau}-1)e^{2\ln S_t + 2\tau\ln(1+r)}. \tag{0.3}$$

So the price of the call option reads

$$\begin{aligned} c_{\text{call}} &= (1+r)^{-\tau} E_Q((S_T-K)^+|\mathcal{F}_t) \\ &= (1+r)^{-\tau}\int_K^\infty (S_T-K)dF(S_T) \end{aligned}$$

where $dF(S_T)$ denotes the lognormal distribution for $S_T$ with mean and variance computed before. We now need to recall a few properties of the lognormal distribution: given a normal random variable $Y \sim N(\nu,\rho^2)$, then $X = e^Y$ is lognormal with mean

$$E[X] = e^{\nu+\rho^2/2} \tag{0.4}$$

and variance

$$Var[X] = (e^{\rho^2}-1)e^{2\nu+\rho^2}. \tag{0.5}$$

Moreover, the probability density is

$$dF(x) = \frac{dx}{\nu x\sqrt{2\pi}} \exp\Big(-\frac{1}{2}\Big(\frac{\ln x - \nu}{\rho}\Big)^2\Big)$$

and the cumulative function is

$$F(x) = \Phi((\ln x - \nu)/\rho)$$

where $\Phi(y)$ is the cumulative of a standard normal distribution, i.e.

$$\Phi(y) = \frac{1}{2\pi}\int_{-\infty}^y e^{-t^2/2}dt.$$

We are interested in the expected value of $X$ conditioned on $X > K$ which is

$$L_X(K) := \int_K^\infty \frac{dx}{\nu\sqrt{2\pi}} \exp\Big(-\frac{1}{2}\Big(\frac{\ln x-\nu}{\rho}\Big)^2\Big) = \exp(\nu+\rho^2)\Phi\Big(\frac{-\ln K + \nu + \rho}{\rho}\Big)$$

Contrasting (0.4) with (0.2) and (0.5) with (0.3), we have $\nu = \ln S_t + \tau(\ln(1+r) - \sigma^2/2)$ and $\rho = \sigma^2\tau$, so that

$$\begin{aligned} &\int_K^\infty S_T dF(S_T) = L_{S_T}(K) \qquad (0.6) \\ &= \exp(\ln S_t + \tau(\ln(1+r)-\sigma^2/2) + \sigma^2\tau/2)\Phi\Big(\frac{-\ln K + \ln S_t + \tau(\ln(1+r)-\sigma^2/2)+\sigma^2\tau/2}{\sigma\sqrt{\tau}}\Big) \\ &= S_t(1+r)^\tau\Phi(d_1) \end{aligned}$$

where

$$d_1 = \frac{-\ln K + \ln S_t + \tau(\ln(1+r) - \sigma^2/2) + \sigma^2\tau/2}{\sigma\sqrt{\tau}},$$

and

$$\begin{aligned}\int_K^{\infty} dF(S_T) &= 1 - F(K) \qquad (0.7)\\ &= 1 - \Phi\left(\frac{\ln K - \ln S_t - \tau(\ln(1+r) - \sigma^2/2)}{\sigma\sqrt{\tau}}\right)\\ &= 1 - \Phi(-d_2)\\ &= \Phi(d_2)\end{aligned}$$

where

$$d_2 = \frac{-\ln K + \ln S_t + \tau(\ln(1+r) - \sigma^2/2)}{\sigma\sqrt{\tau}}.$$

Collecting together all the terms we get

$$c_{\text{call}} = S_t\Phi(d_1) - K(1+r)^{-\tau}\Phi(d_2). \qquad (0.8)$$

With the same strategy one gets also the formula for the put option:

$$c_{\text{put}} = K(1+r)^{-\tau}\Phi(-d_2) - S_t\Phi(-d_1).$$

4. Let $(X_t : t \in \mathbb{N})$ independent and identically distributed random variables with $P(X_t = 1) = 1 - P(X_t = -1) = p = 1/2$, and $S_t = X_1 + X_2 + \cdots + X_t$. For $a < 0 < b$, where $a, b \in \mathbb{Z}$, consider the random time

$$\tau(\omega) = \inf\{t \in \mathbb{N} : S_t(\omega) \notin (a, b)\}.$$

(a) Show that $\tau(\omega)$ is a stopping time in the filtration $\mathbb{F} = (\mathcal{F}_t : t \in \mathbb{N})$ where $\mathcal{F}_t = \sigma(S_u : u \le t) = \sigma(X_u : u \le t)$.

**Solution:** We need to check that $\{\omega : \tau(\omega) \le t\} \in \mathcal{F}_t$, i.e.

$$\{\omega : \inf\{u \in \mathbb{N} : S_u(\omega) \notin (a, b)\} \le t\} \in \mathcal{F}_t$$

which is true since $S_u \in \mathcal{F}_t$ for $u \le t$.

(b) Show that $S_t$ is a $\mathbb{F}$-martingale and it is square integrable $E(S_t^2) < \infty$ $\forall t$.

**Solution:** First note that $|S_t| \le t$ then it is integrable. Moreover,

$$\mathbb{E}(S_t|\mathcal{F}_{t-1}) = \mathbb{E}(X_t|\mathcal{F}_{t-1}) + S_{t-1} = \mathbb{E}(X_t) + S_{t-1} = S_{t-1}.$$

(c) Show that the stopped process $(S_{t\wedge\tau} : t \in \mathbb{N})$ is a martingale.

**Solution:** $S_t$ is a martingale and a stopped martingale is a martingale, as we have seen in the lectures.

(d) Show that $P(\tau < \infty) = 1$. Hint: you can use the second Borel Cantelli lemma.

**Solution:** Consider the event $\{\omega : S_k(\omega) = k\}$ with $k \geq b - a + 1$ and $P(\{\omega : S_k(\omega) = k\}) = 2^{-k}$. Then the events

$$A_n = \{\omega : S_{nk} - S_{(n-1)k} = k\}$$

are independent and such that $P(A_n) = 2^{-k}$. Observe that

$$\limsup_{n\to\infty} A_n \subseteq \cup_n A_n \subseteq \{\tau < \infty\}.$$

Since $\sum_n P(A_n) = \infty$, then Borel-Cantelli lemma implies that $P(\limsup_{n\to\infty} A_n) = 1$ and then $P(\{\tau < \infty\}) = 1$.

(e) Compute $P(S_\tau = a)$ and $P(S_\tau = b)$. Hint: show that $S_{t\wedge\tau}$
**Solution:** Note that $P(S_\tau = a) = P(\tau_a < \tau_b)$ and $P(S_\tau = a) + P(S_\tau = b) = 1$ where $\tau_a = \inf_t\{S_t = a\}$ and $\tau_b = \inf_t\{S_t = b\}$. By the bounded convergence theorem we get

$$\lim_{t\to\infty} E(S_{\tau\wedge t}) = E(S_\tau) = E[M_\tau(\chi(\tau_a < \tau_b)+\chi(\tau_a > \tau_b))] = P(\tau_a < \tau_b)a+(1-P(\tau_a < \tau_b))b$$

but $E(S_\sigma) = E(S_0) = 0$, then

$$P(S_\tau = a) = P(\tau_a < \tau_b) = \frac{b}{b-a} \quad \text{and} \quad P(S_\tau = b) = P(\tau_b < \tau_a) = -\frac{a}{b-a}$$

(f) Show that the martingale $S_t$ has $\mathbb{F}$-predictable variation $\langle S\rangle_t = t$ which by definition means that

$$M_t := S_t^2 - t$$

is a $\mathbb{F}$-martingale.
**Solution:** We compute the Doob decomposition of $S_t^2$: the predictable part is

$$A_t = \sum_{s=1}^{t} \mathbb{E}(S_s^2 - S_{s-1}^2|\mathcal{F}_{s-1}) = \sum_{s=1}^{t} \mathbb{E}(2X_s S_{s-1} + X_s^2|\mathcal{F}_{s-1}) = t$$

therefore, $S_t^2 - t$ is a martingale.

(g) Show that $E(\tau) < \infty$. hint: $(M_{t\wedge\tau} : t \in \mathbb{N})$ is a martingale, and we have the upper and lower bounds

$$0 \leq n \wedge \tau = S_{n\wedge\tau}^2 - M_{n\wedge\tau}, \text{ where } S_t^2 \leq \max\{a^2, b^2\}\forall t \qquad (0.9)$$

use Fatou lemma for $n \to \infty$.
**Solution:** First, note that, since $n \wedge \tau$ is monotone, we have

$$\tau = \limsup_{n\to\infty}(n \wedge \tau) = \limsup_{n\to\infty}(S_{n\wedge\tau}^2 - M_{n\wedge\tau})$$

Then the Fatou lemma gives

$$\begin{aligned}\mathbb{E}(\tau) &= \limsup_{n\to\infty} \mathbb{E}(n \wedge \tau) = \mathbb{E}(\limsup_{n\to\infty}(S_{n\wedge\tau}^2 - M_{n\wedge\tau})) \qquad (0.10)\\ &\leq \limsup_{n\to\infty} \mathbb{E}(S_{n\wedge\tau}^2 - M_{n\wedge\tau}) = \limsup_{n\to\infty} \mathbb{E}(S_{n\wedge\tau}^2)\\ &\leq \max\{a^2, b^2\},\end{aligned}$$

where we use the martingale property $\mathbb{E}(M_{n\wedge\tau}) = \mathbb{E}(M_0) = 0$.

(h) Compute the expectation $E(\tau)$. Hint compute $E(S_\tau^2)$, and take the expectation in (0.9), and use monotone convergence theorem and Lebesgue dominated convergence theorem.

**Solution:** The monotone converge theorem implies that

$$\lim_{n\to\infty} \mathbb{E}(n \wedge \tau) = \mathbb{E}(\tau)$$

while the dominated convergence theorem implies that

$$\lim_{n\to\infty} \mathbb{E}(S_{n\wedge\tau}^2 - M_{n\wedge\tau}) = E(S_\tau^2 - M_\tau)$$

since $S_\tau^2 - M_\tau \in L^1(\mu)$, being $\mathbb{E}(\tau) < \infty$. Therefore, from (0.9)

$$E(\tau) = E(S_\tau^2 - M_\tau) = E(S_\tau^2) = \mathbb{E}[S_\tau^2(\mathbf{1}(\tau_a \leq \tau_b) + \mathbf{1}(\tau_b < \tau_a))] = |ab|.$$