Introduction to mathematical finance I, Exercise 1

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1. Let V be a vector space, for example $V = \mathbb{R}^d$. A set $\mathcal{C} \subseteq V$ is convex if and only if

$$x, y \in \mathcal{C},\ 0 \leq \alpha \leq 1 \Longrightarrow \alpha x + (1-\alpha) y \in \mathcal{C}$$

Show that for $n \in \mathbb{N}$,

$$x_i \in \mathcal{C},\ \alpha_i \geq 0, i = 1, \dots, n \text{ and } \sum_{i=1}^{n} \alpha_i = 1,$$

$$\Longrightarrow \sum_{i=1}^{n} \alpha_i x_i \in \mathcal{C}$$

Solution: we proceed by induction in n: note that the thesis is trivial when $n = 1, 2$ by the definition of convex set. As induction hypothesis we assume that

$$\sum_{i=1}^{n} \alpha_i x_i = z \in \mathcal{C}$$

where $x_i \in \mathcal{C}$ and $\alpha_i \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$. Let be $x_{n+1} \in \mathcal{C}$ and $0 \leq \beta \leq 1$ and consider $\beta x_{n+1} + (1-\beta) z$ which belongs to $\mathcal{C}$. This means that if we define $\underline{\alpha'} := (\beta\alpha_1, \dots, \beta\alpha_n, (1-\beta))$, then $\sum_{i=1}^{n+1} \alpha'_i x_i \in \mathcal{C}$ since $\sum_{i=1}^{n+1} \alpha_i = 1$, therefore the induction argument is complete.

2. Let A be a $(d \times n)$ matrix, and $b = (b_1, \dots, b_d) \in \mathbb{R}^d$.

Either of these two alternatives always holds:

(a) There is $x = (x_1, \dots, x_n)^\top \in \mathbb{R}^n_+$ such that $j = 1, \dots, n$ jolla $Ax = b$

(b) There is $y = (y_1, \dots, y_d) \in \mathbb{R}^d$ such that $yA \in \mathbb{R}^d_+$ and $b \cdot y < 0$.

Prove Farkas' lemma by using the separating hyperplane theorem.

Hint Think about the geometry of the problem: if $a_1, \dots, a_n \in \mathbb{R}^d$ are the column vectors of the matrix A, you can show that

$$\mathcal{C} = \left\{ \sum_{i=1}^{n} \alpha_i a_i :\ \alpha_i \in \mathbb{R}_+ \right\} \subseteq \mathbb{R}^d$$

which is the convex cone generated by the vectors $a_1, \dots, a_n$, is actually convex and closed in $\mathbb{R}^d$.

and the alternatives (a) and (b) correspond to the cases where $b \in \mathcal{C}$ and $b \notin \mathcal{C}$, respectively.

Solution: We recall that the separating hyperplane theorem guarantees that if C is a closed convex sets in $\mathbb{R}^n$ and $b \notin C$, then there exists a hyperplane strictly separating C and x, i.e. there exists $a \in \mathbb{R}^n$ such that

$$a^\top b > \sup_{x \in C} a^\top x$$

We start by showing that (a) implies $\vdash (b)$: let be $\bar{x} \geq 0$ the solution to the system $Ax = b$, then $yA \geq 0$ and $b^\top \cdot y < 0$ is infeasible because

$$0 > b^\top \cdot y = (Ax)^\top \cdot y = x^\top \cdot (A^\top y) \geq 0.$$

Now we show that $\vdash (a)$ implies (b), then $b \notin \mathcal{C}$. Consider the cone

$$\mathcal{C} = \{\sum_{i=1}^{n} \alpha_i a_i : \alpha_i \in \mathbb{R}_+\} \subset \mathbb{R}^d$$

Assume for a moment that $\mathcal{C}$ is convex and closed, then the separating hyperplane theorem implies that there exists y such that

$$y^\top b > \sup_{c \in C} y^\top c$$

and then

$$y^\top b > \sup_{x \geq 0} y^\top \cdot Ax.$$

Since $0 \in \mathcal{C}$, then $y^\top b > 0$. Furthermore, $yA \leq 0$, since otherwise, if say $(yA)_1 > 0$, then we can find a vector $\bar{x}$ such that $\bar{x}_1 = \beta > 0$ and $\bar{x}_2 = \cdots = 0$, so that

$$\sup_{x \geq 0} yA \cdot x \geq yA \cdot \bar{x} = \beta (yA)_1 \to \infty \text{ as } \beta \to \infty$$

which is absurd because $y^\top b > \sup_{x \geq 0} y^\top \cdot Ax$. Taking $y \to -y$ we get the claim. We are left with showing that the cone is convex and closed. The convexity is easy to see by just checking the definition.

For the closedness, we need to show that if the sequence $z^{(n)} \in \mathcal{C}$, then $\lim_{n\to\infty} z^{(n)} =: z \in \mathcal{C}$. We note that $z^{(n)} = \sum_i^n \alpha_i^{(n)} a_i$ for some $\alpha_i^{(n)} \geq 0$, in case the a_i are not independent we can represent $z^{(n)}$ as $z^{(n)} = \sum_i^k \beta_i^{(n)} a_i$, with $k \leq n$ and $\beta_i^{(n)} \geq 0$, so that the representation of $z^{(n)}$ in terms of $\beta_i^{(n)}$ is unique. Then taking the limit $\lim_{n\to\infty} z^{(n)}$ amounts to take the limit $\lim_{n \to n} \beta_i^{(n)} =: \beta_i$. Since the limit preserves the inequality, then $\beta_i \geq 0$ and the closedness is achieved.

3. Prove Gordon theorem: for a matrix $A \in \mathbb{R}^{d \times n}$,

   either $yA > 0$ for some $y \in \mathbb{R}^d$, ($r = (r_1, \ldots, r_d) > 0$ means $r_i > 0\ \forall i$),

   or $Ax = 0$ for some $x \in \mathbb{R}^n_+ \setminus \{0\}$.

   Solution: let denote by $a_1, \ldots, a_n \in \mathbb{R}^d$ the columns of A and let be $a'_i = [a_i^\top\ 1]^\top$, $b' = [0^d\ 1]^\top$ and A' the matrix whose columns are a'_i. By Farkas' lemma we have either $b' \in cone(a'_1, \ldots, a'_n)$ or there exists $y' \in \mathbb{R}^{d+1}$ such that $y'A' \geq 0$ and $y'b' < 0$. If we set $y' = [y\ s]^\top$ with $s \in \mathbb{R}$, then we can rephrase the implications of Farkas' lemma by saying that either there exists $x \in \mathbb{R}^n_+$ such that $0^d = \sum_i x_i a_i$ (i.e. $0 = Ax$) and $\sum_i x_i = 1$, so $x \neq 0$, or there exist $y \in \mathbb{R}^d$ and $s \in \mathbb{R}$ such that $yA + s1^n \geq 0$ and $s < 0$, which equivalent to the claim.

4. A betting-website offers the following multiplier coefficients for the football game Barcelona-Manchester City:

   1.85 for a Barcelona win, 4.3 for a Manchester city win, 3.5 for a draw,

   Is this pricing system arbitrage free ? Is it possible for a gambler to construct an arbitrage strategy with non-negative bets (without short positions?)

   Solution: From the lecture notes we know that the arbitrage-free condition is

$$p_B + p_M + p_D = 1$$

website	a	b	c	d	e	f	g
Barcelona wins	1.85	1.80	1.95	1.80	1.85	1.85	1.75
Manchester City wins	4.30	4.55	4.35	4.30	4.55	4.60	4.70
Draw	3.50	3.55	3.35	3.70	3.30	3.45	3.55

Table 1: gambling multipliers

where $p_B = Pr(\text{Barcelona wins}), p_M = Pr(\text{Manchester wins}), p_D = Pr(\text{Draw})$ and in our case $p_B = 1/1.85$, $p_M = 1/4.3$ and $p_D = 1/3.5$. Therefore, we have

$$p_B + p_M + p_D \simeq 1.059$$

so the bookmaker will end up gaining money whatever is the outcome.

5. Table (1) shows the coefficients for Barcelona-Manchester-City game offered by 7 different gambling websites:

   Check whether a gambler can find an arbitrage possibility with non-negative bets (without short positions) by using the highest multipliers offered for each result.

   Solution: Applying the same reasoning as for the previous exercise and picking the highest multiplier for each outcome we get

   $$p_B + p_M + p_D \simeq 0.996,$$

   so it is possible to choose a strategy (y_B, y_M, y_D) to gain money independently of the outcome. More precisely, if we want to bet 1 €, we would need to implement the following strategy: $y_B = p_B/(p_B + p_M + p_D), y_M = p_M/(p_B + p_M + p_D), y_D = p_D/(p_B + p_M + p_D)$, so that in any case the gain would be

   $$\frac{1}{p_B + p_M + p_D} - 1 > 0$$