Matrix-free X-ray tomography with sparse data

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INTRODUCTION

Consider the following linear inverse problem: given a vector m and matrix A, find x such that

$$Ax = m, \quad (1)$$

where the matrix A is ill-conditioned, i.e. its condition number is large.

This problem can be studied with tomographic X-ray projection data. In tomographic X-ray imaging our data m consist of X-ray projection images from several different directions and x is the inner structure of the object being scanned. The inverse problem is to find x. This kind of measurement can be written in the form (shown in [1], page 22-29)

$$m = Ax + \varepsilon. \quad (2)$$

This problem is ill-posed. It is very sensitive to measurement noise and modelling errors. In this project we study methods that can solve this problem noise robustly and can be implemented to large datasets.

As data we use a dataset measured in X-ray facility of the Industrial Mathematics Laboratory. We are using 15 projection directions.

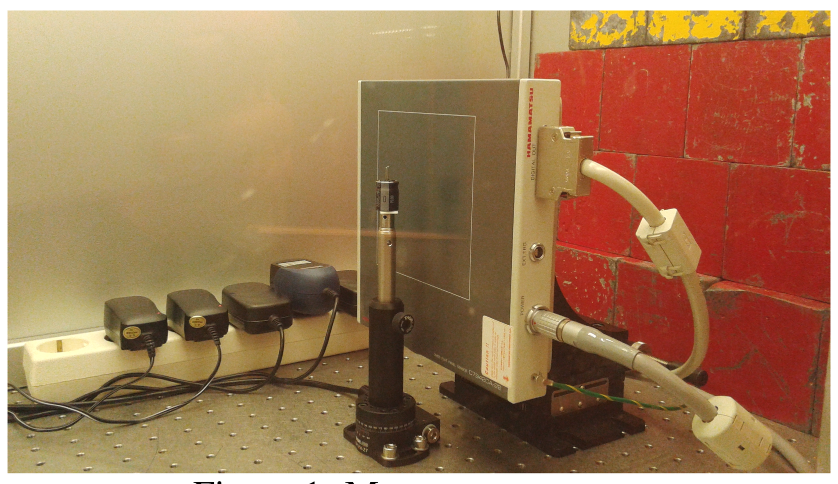

Figure 1: Measurement setup

As can be seen from Fig. 1, our measured object is a capacitor. The goal in this study is to find a good slice picture from the inner structure of this capacitor with only 15 projection angles.

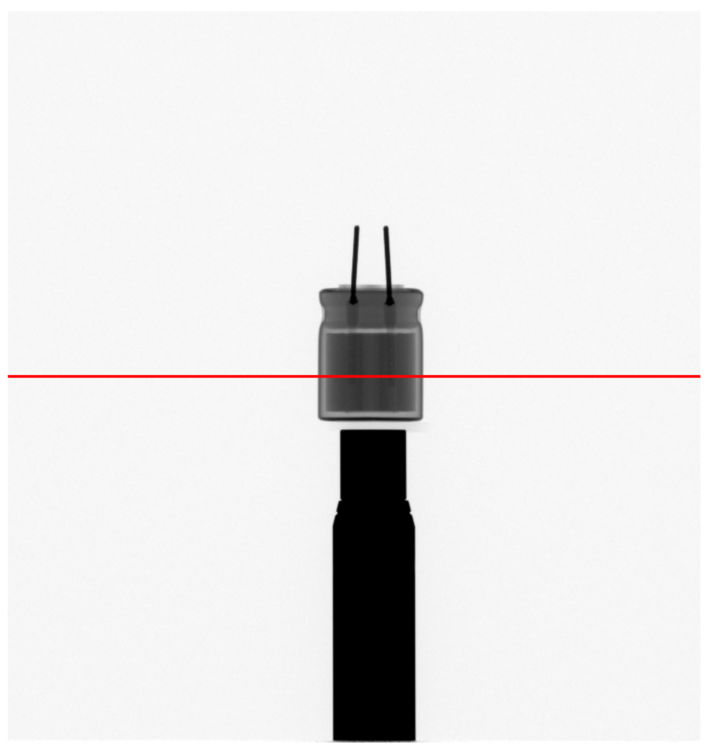

Figure 2: Close-up of the capacitor and the height at which the cross-section will be studied

METHODS AND MATERIALS

Tikhonov regularization

Our method of choice for solving equation (1) is Tikhonov regularization, which is a useful regularization method for solving linear inverse problem. The Tikhonov regularized solution to the equation $Ax = m$ is defined as the vector $T_\alpha(m) \in \mathbb{R}^n$ that minimizes the expression

$$\|AT_\alpha(m) - m\|^2 + \alpha \|T_\alpha(m)\|^2,$$

where $\alpha > 0$ is a regularization parameter. It is evident from the definition that the Tikhonov regularized solution is a balance between a small residual $AT_\alpha(m) - m$ and a small solution $T_\alpha(m)$.

As proven in [1] (equation (5.9), page 67), the Tikhonov regularized solution is given by

$$T_\alpha(m) = \left(A^T A + \alpha I\right)^{-1} A^T m. \quad (3)$$

It is to be noted, however, that Tikhonov's method does not in itself provide a way to determine the optimal value of the regularization parameter α. For this purpose, we will be using the L-curve method.

L-curve method

In the L-curve method, we choose a collection of different regularization parameters and compute the Tikhonov regularized solution $T_\alpha(m)$ for each of them. Since Tikhonov regularization is an attempt to find a balance between the terms $\|AT_\alpha(m) - m\|$ and $\|T_\alpha(m)\|$, the points

$$(\log \|AT_\alpha(m) - m\|, \log \|T_\alpha(m)\|) \in \mathbb{R}^2 \quad (4)$$

are plotted. The resulting curve will usually resemble the letter "L". The optimal value of α is expected to be found as close to the corner of the "L" shape as possible.

However, explicitly computing the matrices in equations (3) and (4) would be cumbersome, and so we will use a matrix-free iterative method called the conjugate gradient method instead.

Conjugate gradient method

Conjugate gradient method is used for solving Quadratic optimization problems

$$minimize\ \frac{1}{2}\mathbf{x}^T H\mathbf{x} - \mathbf{b}^T\mathbf{x}, \quad (5)$$

where H is an $n \times n$ known symmetric positive definite matrix.

For finding the minimum we use n conjugate directions $d_1, d_2, \ldots, d_n$, such that progress made in one direction does not effect progress made in other directions. The solution is reached in k steps, where $k \leq n$. In conjugate gradient method the next direction is determined in each iteration (using conjugate Gram-Schmidt process [3]). The first direction is a Steepest decent step.

Consider the formulation (3) of Tikhonov regularization

$$(A^T A + \alpha L^T L)\mathbf{f} = A^T\mathbf{m}, \quad (6)$$

where $\alpha A^T A = \alpha I$.

Let's denote $H = A^T A + \alpha I$ and $\mathbf{b} = A^T\mathbf{m}$. Now we can form a quadratic optimization problem

$$minimize\ \frac{1}{2}\mathbf{f}^T H\mathbf{f} - \mathbf{b}^T\mathbf{f} \quad (7)$$

where Hf can be computed matrix-free with radon-function.

In our MATLAB implementation we will use the algorithm introduced in [2].

RESULTS

Measurement results

The measurements of the X-ray projections (using all 180 measurement angles) can be seen in sinogram form in Fig. 3.

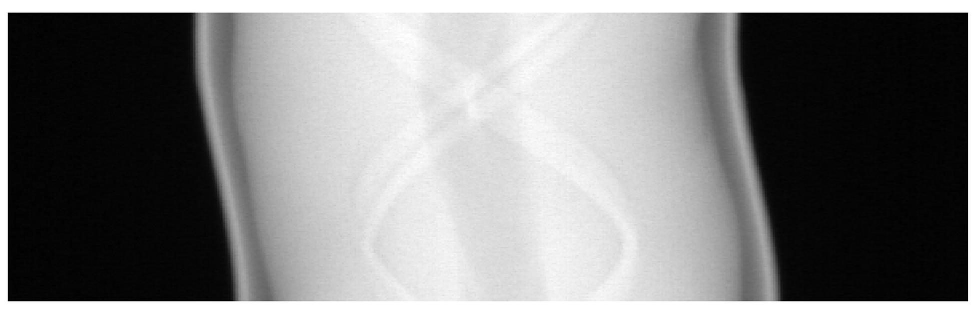

Figure 3: The X-ray measurement data as a sinogram

L-curve method

The L-curve method yielded the value $\alpha = 1.1316$ as the optimal value of the regularization parameter α. The plot of the L-curve can be seen in Fig. 4.

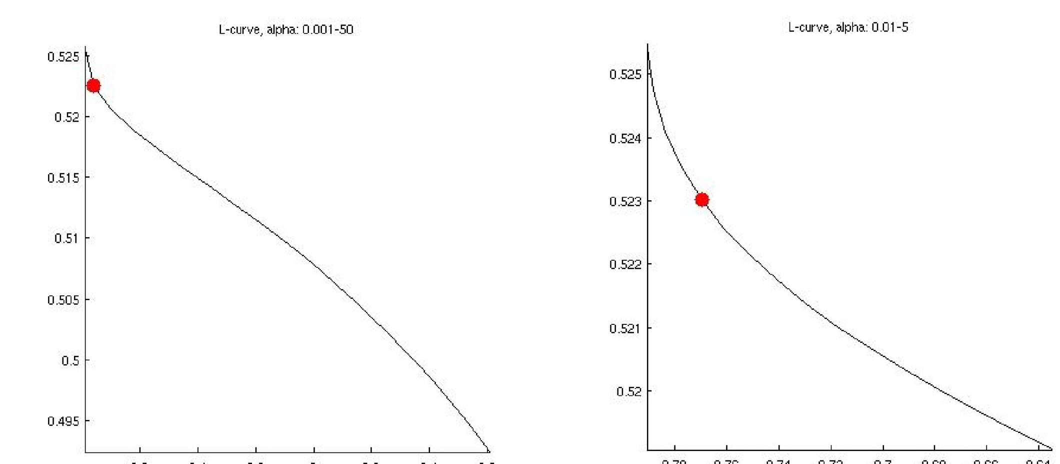

Figure 4: L-curve and L-curve focused in a smaller area

Tikhonov regularized reconstruction

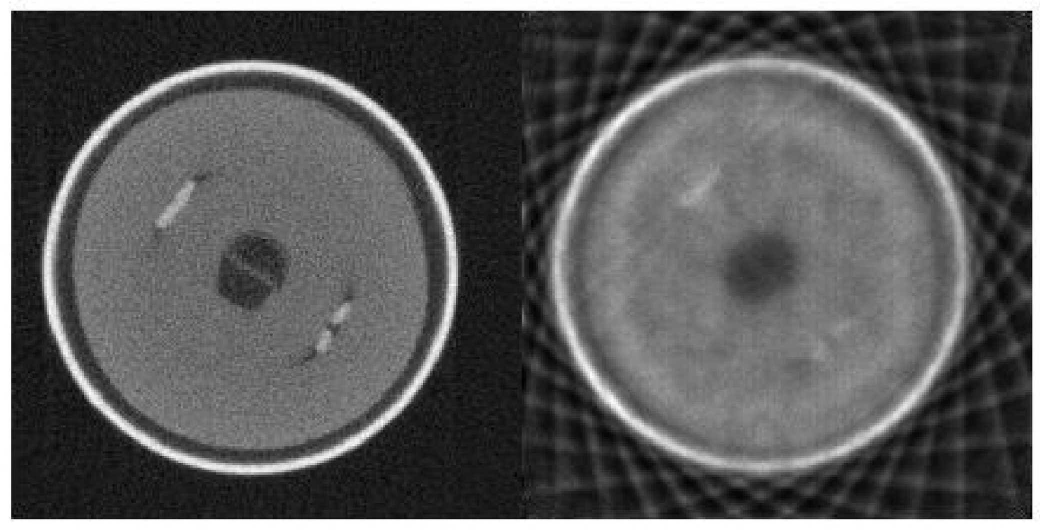

Figure 5: Tikhonov regularized reconstruction

Using the optimal regularization parameter given by the L-curve method and 15 measurement angles, Tikhonov regularization yielded a reconstruction with a relative error of 29% when compared to the reconstruction given by using 180 angles. Reconstructions with 180 angles and 15 angles can be seen side by side in Fig. 5.

DISCUSSION

As can be seen from Fig. 5, the reconstruction yielded by Tikhonov regularization using 180 measurement angles was excellent. Furthermore, even when only using 15 angles, the reconstruction still preserved many of the detailed features of the measured object. However, since Tikhonov regularization is not an edge-preserving method, the reconstruction is somewhat blurry. Using an edge-preserving method (such as total variation regularization) would possibly have yielded a sharper – though not necessarily *better* – reconstruction.

For comparison, we also used filtered back-projection on our dataset to see how well it would perform with sparse measurement angles. For this, we used MATLAB's built-in `iradon.m` function with the same 15 measurement angles as in our Tikhonov reconstruction. The resulting reconstruction can be seen in Fig. 6.

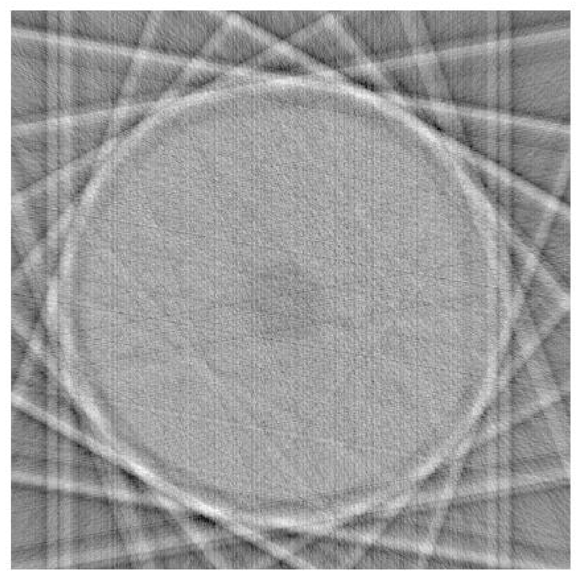

Figure 6: Reconstruction using filtered back-projection with 15 measurement angles

As can be seen from Fig. 4, the L-curve does not have a sharp L-shaped curve. So it is non-trivial to choose the right parameter. A smooth L-curve actually characterizes the real situation quite well. There was no significant difference between reconstructions with different alpha values until the L-curve started to decline faster again (Fig. 4).

References

[1] Mueller J L and Siltanen S, *Linear and Nonlinear Inverse Problems with Practical Applications*, SIAM 2012.

[2] C.T. Kelley, *Iterative Methods for Optimization*, SIAM, 1999, page 7.

[3] Jonathan Richard Shewchuk, *an Introduction to the Conjugate Gradient Method Without the Agonizing Pain Edition 1* August 4, 1994, page 25,31