

# X-Ray tomography with sparse data

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## INTRODUCTION

Consider the linear model

$$\mathbf{m} = A\mathbf{f} + \varepsilon \quad (1)$$

where  $\mathbf{m} \in \mathbb{R}^k$  is the measured data,  $\mathbf{f} \in \mathbb{R}^n$  is the target of our interest,  $A \in \mathbb{R}^{k \times n}$  is the matrix connecting the data and the quantity of interest and  $\varepsilon \in \mathbb{R}^k$  represents the errors coming from the noise mainly caused by electrical disturbances in the measurement device.

In this study we are dealing with X-ray tomography with limited data, which means that we measure certain object (a seed of a peach in our case) with suitable X-ray device, in this case located in Helsinki University, Department of Physics in Kumpula, from only a few angles. From measurements we get a collection of X-ray images of our object,  $\mathbf{m}$  of our model (1) but the interest of our study is to find out what is inside the object. To find this out we have to solve  $\mathbf{f}$  which gives us an approximate reconstruction of the inner structure of the object at hand.



Figure 1: A seed of a peach.

We are dealing with an inverse problem:

Given noisy measurement  $\mathbf{m} = A\mathbf{f} + \varepsilon$ ,  
find out information about  $\mathbf{f}$ .

Inverse problems are hard problems to solve and we usually can't find the real solution but luckily there are methods to get good results. The problems lie in finding a unique solution that is continuously dependent on the data. One of the most standard methods and also the method of our choice is the Tikhonov regularization. This might not be the best method for our purpose but choosing the regularization parameter optimally we can hope to get the best possible results available. In this study we are going to use the L-curve method to find out the optimal parameter value.

Finally, since the data turned into matrix form can be quite large we do not want to construct all the matrices involved that are also vulnerable to computational errors, we use an iterative method, namely the Conjugate gradient method (cg), to find the best reconstruction possible.

## METHODS AND MATERIALS

### • Tikhonov regularization

The Tikhonov regularized solution of equation (1) is the vector  $T_\alpha(\mathbf{m}) \in \mathbb{R}^n$  defined by

$$T_\alpha(\mathbf{m}) = \arg \min_{\mathbf{z} \in \mathbb{R}^n} \left\{ \|A\mathbf{z} - \mathbf{m}\|^2 + \alpha \|\mathbf{z}\|^2 \right\}, \quad (2)$$

where  $\alpha > 0$  is the regularization parameter. We see that the solution  $T_\alpha(\mathbf{m})$  is a balance between the two requirements:

- (1) The residual  $AT_\alpha(\mathbf{m}) - \mathbf{m}$  should be small.
- (2) Solution  $T_\alpha(\mathbf{m})$  should be small in  $L_2$ -norm.

The Tikhonov regularized solution  $T_\alpha(\mathbf{m})$  satisfies the normal equations

$$(A^T A + \alpha I)T_\alpha(\mathbf{m}) = A^T \mathbf{m}. \quad (3)$$

Writing  $\tilde{A} = A^T A + \alpha I$ , we minimize (3) by minimizing

$$f^T H f - 2b^T f, \quad (4)$$

where  $H = \tilde{A}^T \tilde{A}$  is symmetric, positive-definite matrix and  $b^T = \mathbf{m}^T A \tilde{A}$ . We have arrived in a form that of quadratic optimization problem. For such problem we can use the conjugate gradient method.

### • Conjugate gradient method

Method is an iterative solution method for objects of form (4) where  $H$  is a spd matrix. We end up minimizing residual  $r = b - Hf$ , where  $b$  contains the backprojected data. Matrix  $Hf$  is constructed by  $f_0$  and regularization parameter  $\alpha$ .

We repeat steps (1) and (2) for prefixed number of iterations until residual  $r$  is sufficiently small:

- (1) Noting that residual  $r$  is the negative gradient at  $f = f_k$ , it is the first direction that we want to go.
- (2) By step (1) we get point  $f_{k+1}$  and new residual  $r$ .

On output we get the approximate solution  $f$ .

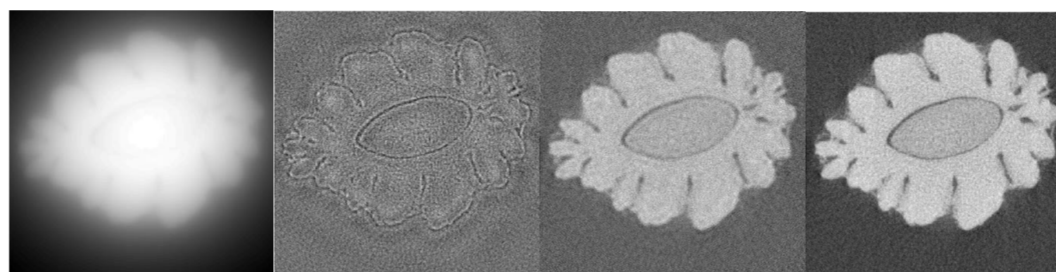


Figure 2: The reconstruction transforms as the amount of iterations is changed. In the leftmost picture only 1 iteration is used. The reconstruction gets better when the amount of iterations is increased, rightmost picture is with 80 iterations.

### • L-curve method

Suppose that we have a collection of regularization parameters:

$$0 < \alpha_0 < \alpha_1 < \dots < \alpha_m < \infty.$$

To choose the best value, for every  $\alpha_i$  we calculate the point

$$(\log \|\mathbf{m} - R(\mathbf{x})\|, \log \|\mathbf{x}\|) \in \mathbb{R}^2,$$

where  $\mathbf{x}$  is the regularized solution calculated by conjugate gradient method and  $R$  denotes the radon transform. Next we plot these points and look for the optimal value for  $\alpha$  which is found near the corner of the curve. See Figure 7.

## RESULTS

Using the methods presented above we recovered cross-sections of our object. We fixed a row, close to the middle of the seed, from our original images obtained from the measurements and reconstructed slices from that exact part. In the measurement session we took pictures from 180 different angles, but we only used a sparse set of these images for our reconstructions as seen in Figure 3 - Figure 5. For comparison we also reconstructed the cross-section with the whole data set (Figure 6).

The values of the regularization parameters used varied for different set of images used. For 12 angles the parameter was  $\alpha = 6$  being the smallest and for 30 angles  $\alpha = 7.94$ .

As the Conjugate gradient method is an iterative solution method it requires a choice of the amount of iterations. In Figures 3-6 we used 80 iterations.

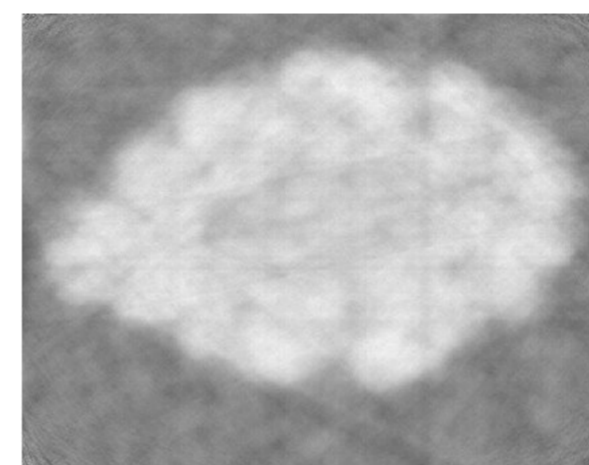


Figure 3: Reconstruction with 12 projection angles.

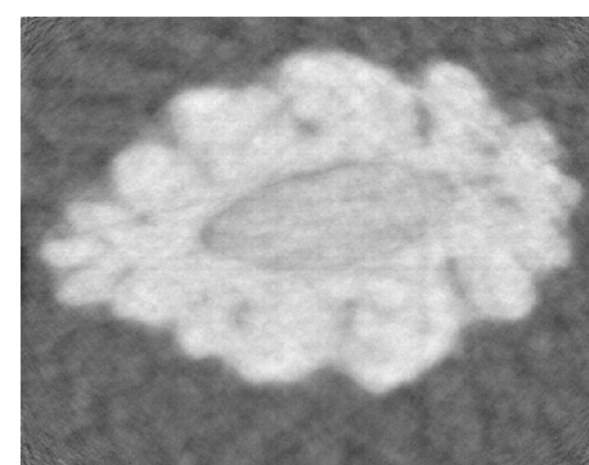


Figure 4: Reconstruction with 20 projection angles.

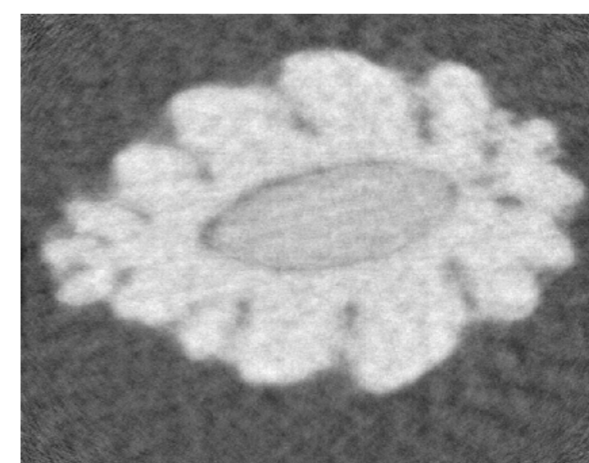


Figure 5: Reconstruction with 30 projection angles.

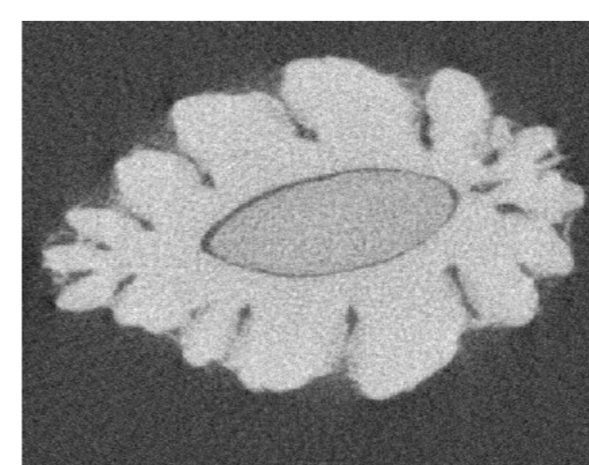


Figure 6: Reconstruction with 180 projection angles.

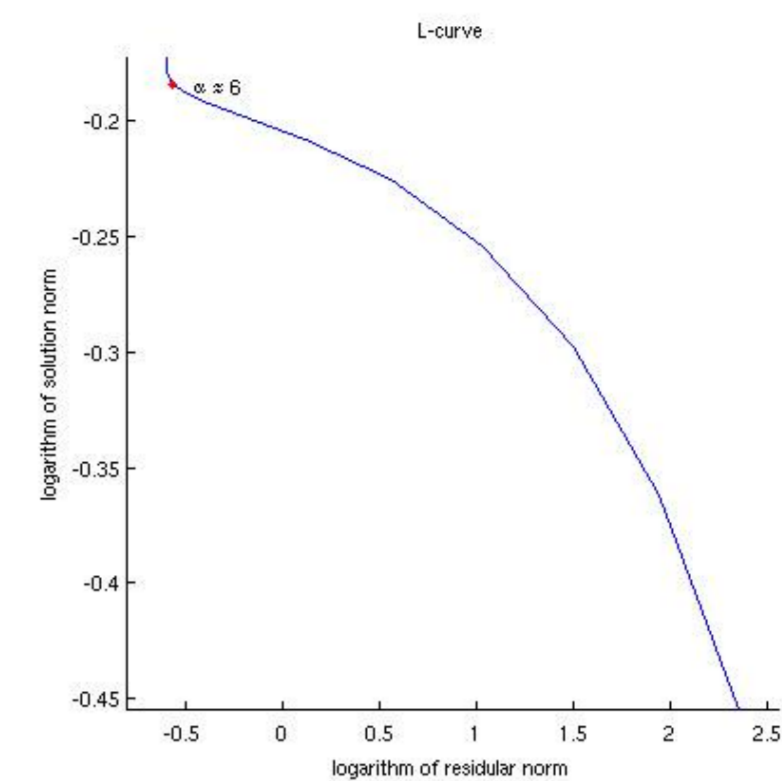


Figure 7: The L-curve with 12 projection angles. Red dot marks the optimal value of  $\alpha$ .

The L-curve method was used manually. This means that we looped over multiple values of the regularization parameter  $\alpha$  ranging from  $10^{-10}$  to  $10^3$ , plotted the values and searched for the optimal value as near as the corner of the curve as possible. Figure 7 demonstrates a typical L-curve that we were able to produce with 12 projection angles. For different values of images used the L-curves looked essentially similar.

## DISCUSSION

Our goal in this study was to find out how Tikhonov regularized solution method works in practice with sparse data. Using as little as 12 angles for the reconstruction gives relatively poor results but it is still better than the standard inverse radon transform, 'filtered-back projection', as can be seen by comparing Figures 3 and 8.

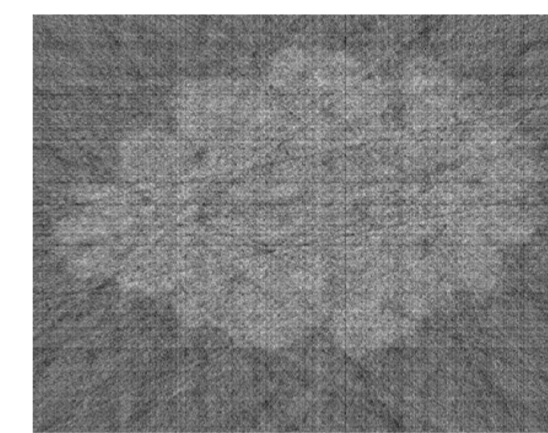


Figure 8: The Filtered-back projection of data set consisting 12 images.

Adding a few more images to the set from which the reconstruction is to be done improves the quality of the reconstruction significantly. With 30 projection angles smaller details of the object become visible and the reconstruction can be considered quite good. This interpretation is of course very relative and does not apply to more sensitive situations like the imaging the head of a human. With larger amounts of angles the computing time increases rapidly, which is not preferable from the practical point of view. However the results are not that much better than with the simple filtered-back projection.

The L-curve method seemed to give good values for the regularization parameter. Calculating the regularized solution with smaller or bigger values than the value given by the L-curve resulted in worse images than with the optimal value. This judgement was naturally done by our own human perception, which can be deceiving, since no relative errors or other quantities related to the successfulness of the solution could not be calculated.

## References

- [1] Mueller J L and Siltanen S, *Linear and Nonlinear Inverse Problems with Practical Applications*, SIAM 2012.
- [2] Kelley, C.T., *Iterative Methods for Optimization*, SIAM 1999.