Inverse Problems Project: Pistachio

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Introduction

In this poster we present sparse tomography reconstructions of a pistachio nut.

In a *sparse-angle imaging problem* like we consider in this study, projections are available only from a limited amount of projection angles. Creating a two-dimensional reconstruction image from such limited data is a non-trivial inversion problem, and thus warrants use of regularized inversion methods like the Total Variation regularization method used here. Regularization parameter α was chosen automatically by a method based on comparing TV norms of reconstructions with different values of α.

The x-ray imaging of the test object was done in the Industrial Mathematics Laboratory at University of Helsinki with the assistance of Alexander Meaney.

Materials and Methods

Linear Inverse Problem Model and X-ray imaging

Generally this kind of imaging problem can be modelled as

$$\mathbf{m} = A\mathbf{f} + \varepsilon, \tag{1}$$

where $\mathbf{m}$ is vector of measurement data, $\mathbf{f}$ is finite-dimensional approximation vector modeling the target function f we want to reconstruct, ε is random noise and matrix A is a linear operator. In a tomographic problem, the attenuation function f describes a two-dimensional image and A the X-ray projection process (the Radon transform).

The X-ray source used was a 50 kV molybdenum tube (operated at current of 1 mA) manufactured by Oxford Instruments. Projections were taken from 0° to 179° with angular intervals of 1° and exposure time of 1000 ms.

Most X-ray sources, like the one used in this project, produce a multispectral beam (so-called 'fan-beam'), and line integrals of f may differ depending on the propagation direction of the X-ray along the line. However, in this study we use parallel-beam geometry model to simplify computations. Assuming the target is very small and placed close to the detector, we can approximate parallel-beam case even with a fan-beam X-ray source.

To obtain the sparse angle data, instead of doing real sparse angle imaging, the object was imagined from large number of angles, and small number of them were picked for reconstruction to simulate sparse-angle imagined data.

For an interesting and suitably small imaging target for the study, we chose a pistachio nut (Figure 1).

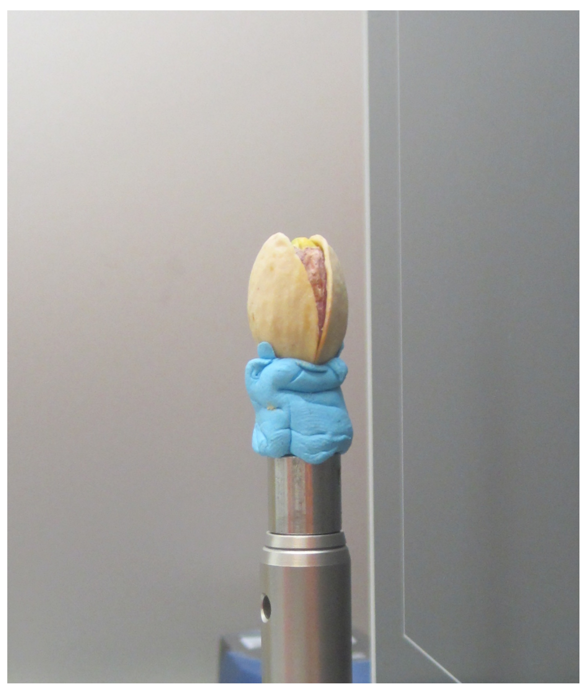

Figure 1: Picture of the target ready for imaging. Size of nut: 1,9 cm long and 1,4 cm wide.

Total variation regularization

Following [5, 4], we define total variation for two-dimensional images as follows:

Definition 1 (Total variation)**.** *Let f be a real-valued differentiable function defined on bounded open set $\Omega \subset \mathbb{R}^2$. Total variation of f can be defined as*

$$TV(f) = \int_\Omega |\nabla f(x)| dx. \tag{2}$$

Replacing TV with its discretized approximation, the total variation regularized solution $T_\alpha(\mathbf{m})$ to inverse problem 1 can now be defined as

$$T_\alpha(\mathbf{m}) = \underset{z \in \mathbb{R}^n}{\arg\min} \left\{ \|A\mathbf{z} - \mathbf{m}\|_2^2 + \alpha \|L\mathbf{z}\|_1 \right\}, \tag{3}$$

where L is a discretized differential operator and $\|\cdot\|_1$ ℓ^1-norm.

From equation (2) we see that minimization of TV should result in a solution with less oscillations characterized by absolute value of the gradient $\nabla f(x)$.

In the numerical implementation, the objective function to be minimized becomes

$$G(\mathbf{f}) = \|A\mathbf{f} - m\|_2^2 + \alpha \sum_{i=1}^{n} \sum_{j=1}^{n} |\mathbf{f}_{i,j} - \mathbf{f}_{i-1,j}| + |\mathbf{f}_{i,j} - \mathbf{f}_{i,j-1}|, \tag{4}$$

where $\mathbf{f}$ is our two-dimensional image.

Numerical implementation with Barzilai-Borwein method

Minimizing the objective function (4) is a numerical optimization problem. For problems where the objective function has a known, continuous gradient that can be efficiently computed, many fast gradient-based optimization algorithms are available, one of them the Barzilai-Borwein (B-B) method.

However, G isn't continuously differentiable, so to use Barzilai-Borwein, we *approximate* the absolute value function $|\cdot|$ in (4) by $|\cdot|_\beta$,

$$|x| \approx |x|_\beta = \sqrt{x + \beta}, \text{ where } \beta > 0 \text{ small,}$$

which naturally has a continuous derivative, and so in numerical calculations we replace (4) with an approximation $G_\beta(\mathbf{f}) :\approx G(\mathbf{f})$. The gradient $\nabla G_\beta(f)$ can now be analytically solved (see [4, Eq 6.13 – 6.14]),

$$\nabla G_\beta(f) = 2A^T A\mathbf{f} - 2A^T \mathbf{m} + \left[\frac{\mathbf{f}_{i-1} - \mathbf{f}_i}{((\mathbf{f}_{i-1} - \mathbf{f}_i)^2 + \beta)^{1/2}} + \frac{\mathbf{f}_{i+1} - \mathbf{f}_i}{((\mathbf{f}_{i+1} - \mathbf{f}_i)^2 + \beta)^{1/2}} \right]_{i=1,\dots,n.} \tag{5}$$

The B-B method (presented in [1]) is a version of the famous iterative *gradient descent* or *steepest descent* algorithm, and it can be described as follows:

After some initial guess $f^{(1)}$, each iteration step $f^{(n)}$ is determined from the previous step

$$f^{(n+1)} = f^{(n)} - \delta_n \nabla G_\beta(f^{(n)}), \tag{6}$$

where the *steplength parameter* δ_n is (in B-B method) given by

$$\delta_n = \frac{y_n^T y_n}{y_n^T g_n}, \tag{7}$$

where $y_n = f^{(n)} - f^{(n-1)}$ and $g_n = \nabla G_\beta(f^{(n)}) - \nabla G_\beta(f^{(n-1)})$.

Automatic regularization parameter choice

In any implementation of total variation regularization, an important question is how to choose the regularization parameter α. In this project, we tested an automatic TV norm based method inspired by the sparsity *S-curve* method proposed in [4, Section 6.3] and [2, 3].

Our 'measurement of sparsity' is simply the TV norm of the (reconstructed) slice image: we noticed that reconstructions with very small α have quite much 'variation' measured by $TV(\mathbf{m})$, and likewise with very large α, $TV(\mathbf{m})$ is small; moreover, curve seems to decrease monotonically (at least in a feasible range). Assuming we have *a priori* knowledge of the desired level of total variation in a good reconstruction, we can find a good guess for α with an interpolation method similar to the 'S-curve':

We compute the reconstructions and their TV norms for multiple but computationally feasible number k of discrete points $\alpha_1, \dots, \alpha_k$ in some range. Our automatic guess is the point α_s that has an interpolated (piecewise cubic interpolation) value nearest to the *a priori* known TV value. (See Figure 4.)

To create *a priori* data similar enough to sparse B-B reconstructions, we used full-angle reconstruction obtained by simple filtered back-projection (FBP) with some noise reduction in addition basic X-ray image preprocessing: Reconstruction image was first filtered using MATLAB's wiener2.m -routine, and then still noisy background was averaged. See Figures 3 for resulting *a priori* data.

The FBP couldn't itself be used because full angle FBP (Figure 2) reconstructed more imaging noise (thus variation) than the sparse reconstructions with 15 – 30 angles (e.g. 6).

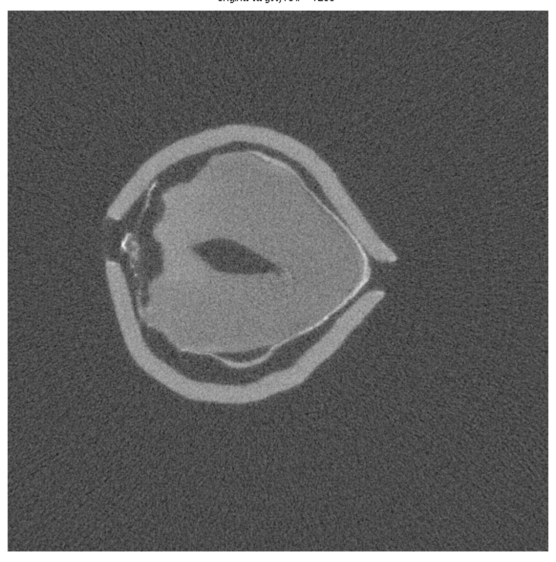

Figure 2: Full-angle reconstruction of the middle part of the target using Filtered Back-projection.

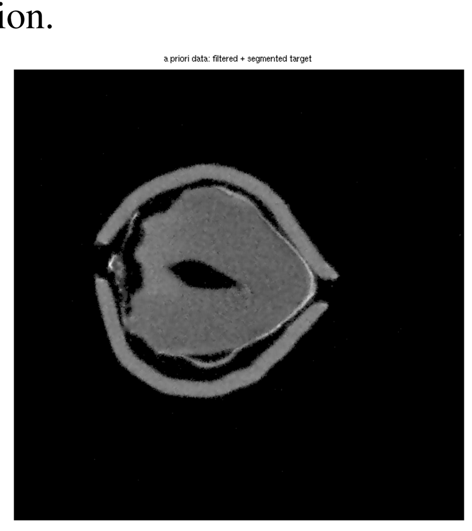

Figure 3: *a priori* data from middle part of the target.

Results

We reconstructed horizontal slices of the target pistachio at two different heigths: from the middle and 'top' level of a nut (in a 'standing position', as in Figure 1). The sparse-angle reconstructions were done with sets of both 30 and 15 even-spaced angles (angles $0^\circ, 6^\circ, \dots, 174^\circ$ and $0^\circ, 12^\circ, \dots, 168^\circ$, respectively). See Figures 5,6 for the resulting reconstructions.

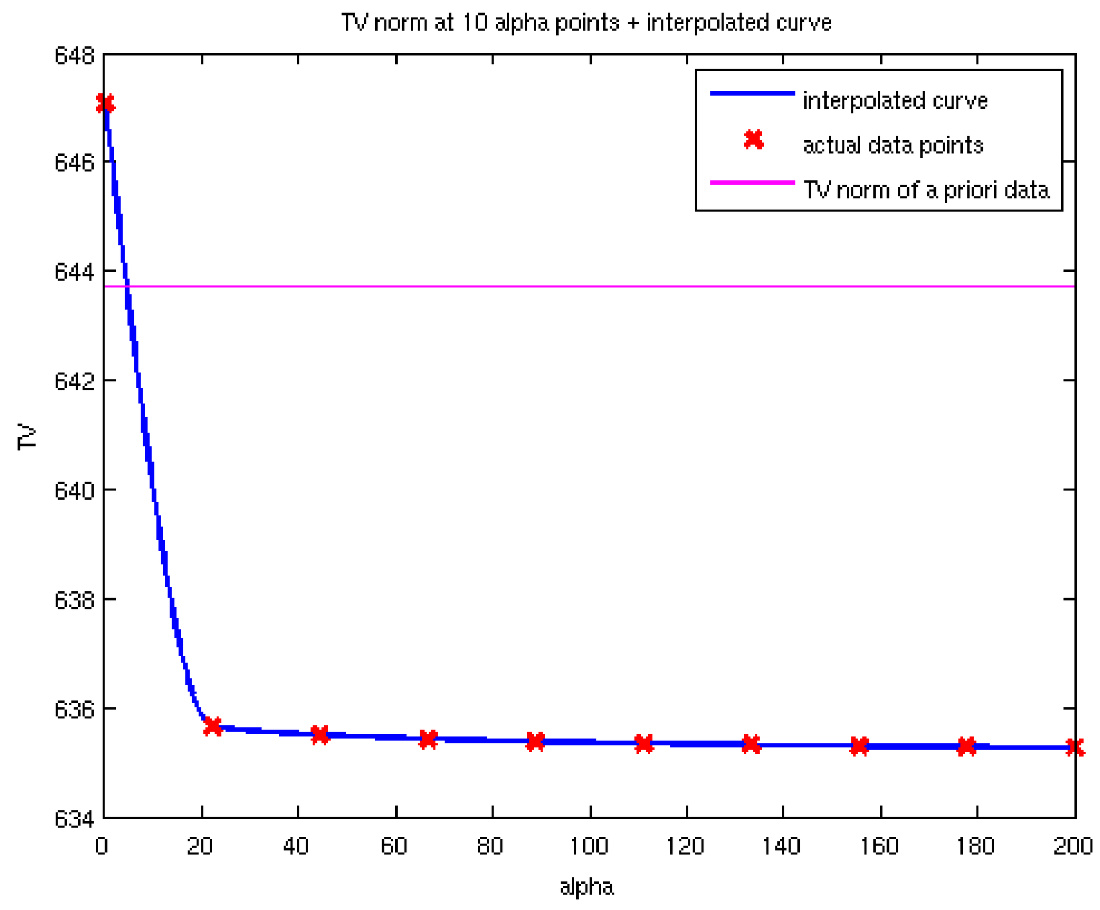

Figure 4: Interpolated (middle section of nut) automatic choice α_s is 4.4186.

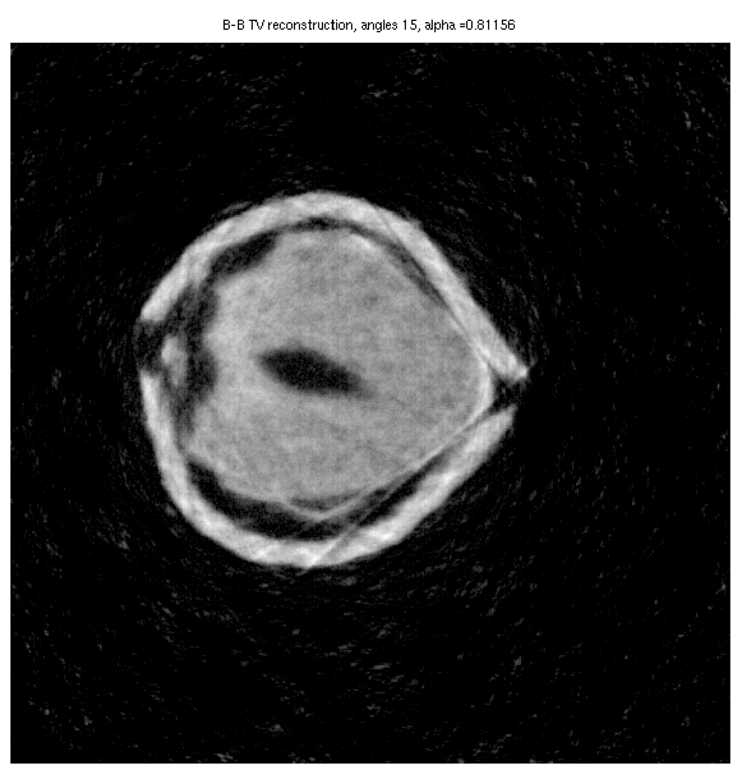

Figure 5: Reconstruction of the middle part by using 15 angles.

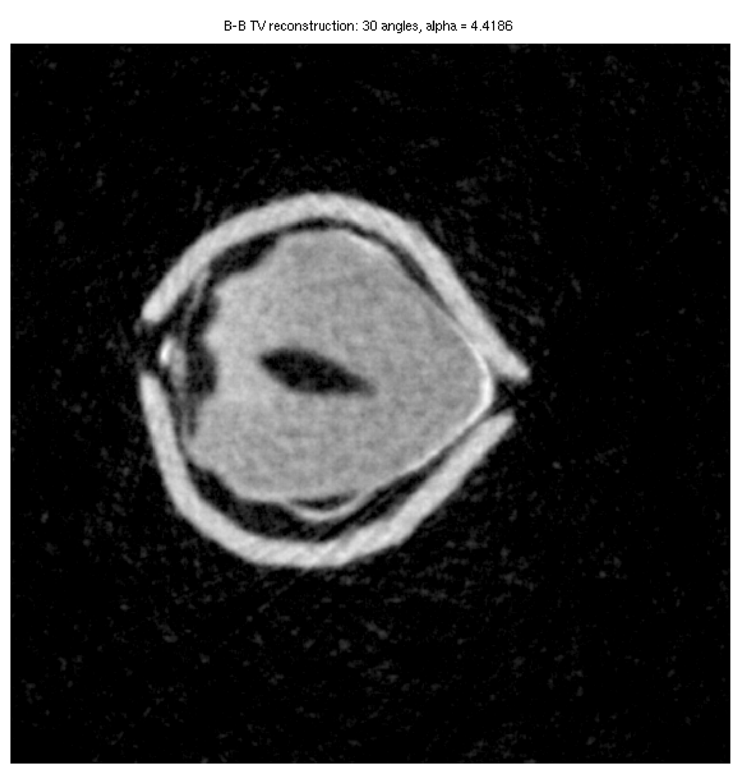

Figure 6: Reconstruction of the middle part by using 30 angles.

Discussion

Reconstructions with both 30 and 15 angles turned out surprisingly good, 30 slightly but noticifiably better. TV managed to reconstruct even the small details clear and sharp, but in the 15 angles version some artifacts were shown.

Even though results are not fully reliable since our *a priori* data was computed from the same full-angle data set than actual reconstructions, this study gives a good demonstration of TV as a tomography reconstruction method and points out TV's power to create sharp, well-detailed reconstructions.

References

[1] J. Barzilai and J. Borwein. "Two-point step size gradient methods". *IMA Journal of Numerical Analysis* 8 (1988).

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[5] Leonid I Rudin, Stanley Osher, and Emad Fatemi. "Nonlinear total variation based noise removal algorithms". *Physica D: Nonlinear Phenomena* 60.1 (1992), pp. 259–268.