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Tomographic Reconstruction using NURBS and MCMC

Zenith Purisha - Prof. Samuli Siltanen

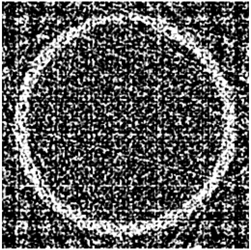
Student Seminar UH - January 30, 2015

Department of Mathematics and Statistics

Target

Target

FBP



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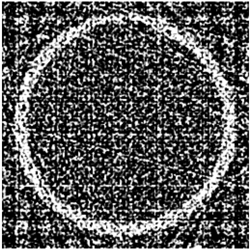
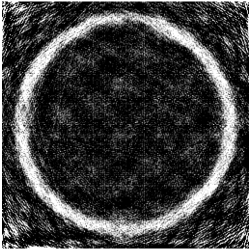


Tikhonov



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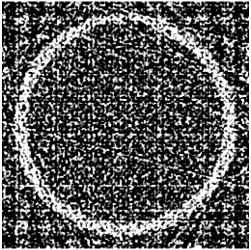

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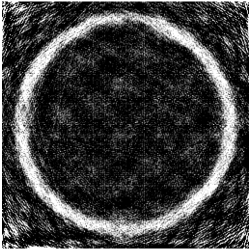
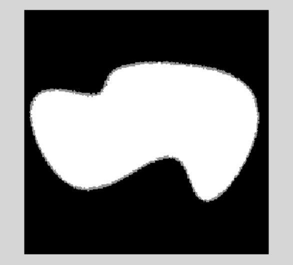
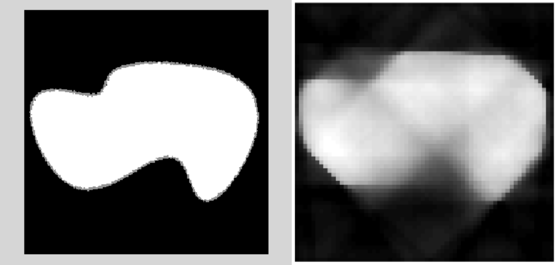


NURBS-MCMC

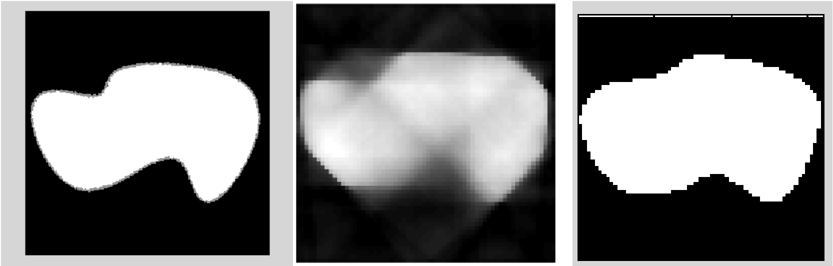
Original



Original, TV



Original, TV and NURBS-MCMC reconstruction



Outline

- Introduction
- Background
  - NURBS
  - Tomographic Measurement Model
  - Bayesian inversion
  - MCMC
- Sugar Reconstruction
- Corrosion Pipe Reconstruction
- Conclusion
- Revisited :NURBS

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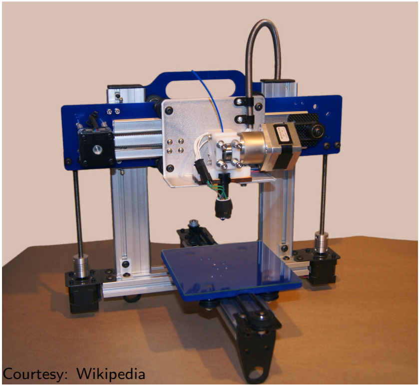
CAD (Computer-Aided Design)

NURBS (Non Uniform Rational B-Splines) the standard tool to represent geometry in CAD systems, have been the building blocks of CAD modelling.

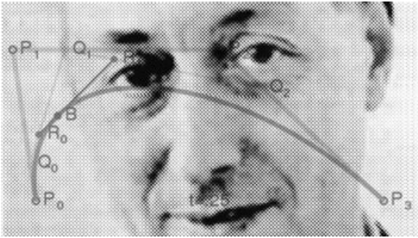
3D-Modelling


Courtesy: Wikipedia

CNC (Computer Numerical Control) system, highly automated using CAD and CAM (Computer-Aided Manucfacturing).

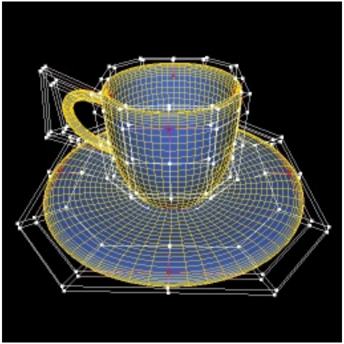
CAD is using NURBS

Why?



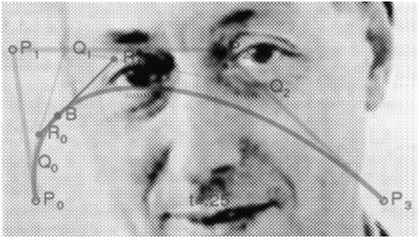
Courtesy: www.aiblog.it

Early 1970s, Pierre Bezier



CAD is using NURBS

Why? **Fast in computation (small parameters)**



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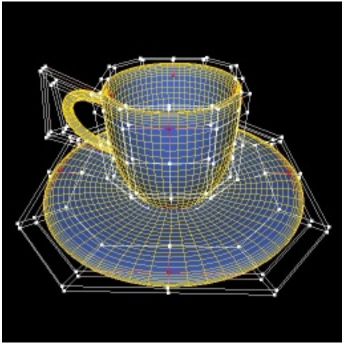
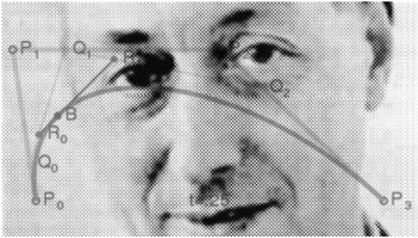


CAD is using NURBS

Why? **Fast in computation (small parameters)**
Efficient!



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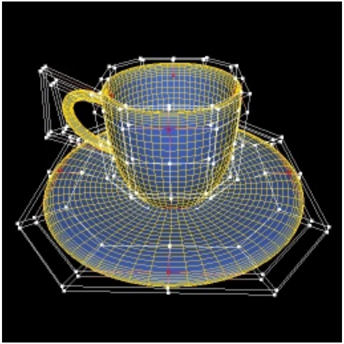

Homogenous Object

- Take the transversal slice from the object.

- Collect the X-ray projection data.
  In other words, we have access to a collection of line integrals of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} c & \text{for } (x,y) \in \Omega, \\ 0 & \text{for } (x,y) \in \mathbb{R}^2 \setminus \Omega. \end{cases} \tag{1}$$

- The angular sampling of the X-ray data is very sparse, allowing for quick measurement process (low radiation dose/ few angle data).
- Our aim is to recover two things: the boundary $\partial\Omega \subset \mathbb{R}^2$ represented as a parameterized curve and the attenuation coefficient c.

Implementing Bayesian Inversion and NURBS in Tomography Reconstruction

Bayesian Inversion and NURBS

$\Downarrow$

Recovering parameters

(Control Points and attenuation value)

Overview

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NURBS curve

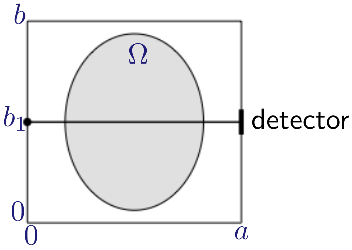
(Loading video)

Video is taken from http://geometrie.foretnik.net

Overview

X-ray Measurement

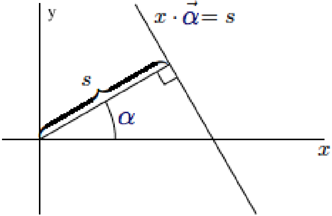
Consider an attenuation function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) \geq 0$ and $\text{supp}(f) \subset \Omega$ with bounded $\Omega \subset \mathbb{R}^2$.



$$\frac{dI(x)}{I(x)} = -f(x, b_1)dx,$$

where $I(x)$ is the intensity of the X-ray at the point (x, b_1) while passing through the source to the detector.

Radon Transform



The radon function of the function f depends on the angular parameter α and on a linear parameter $s \in \mathbb{R}$ as follows:

$$\mathcal{R}f(s,\alpha) = \int_{\mathbf{x}.\vec{\alpha}=s} f(\mathbf{x})d\mathbf{x}^{\perp},$$

where $d\mathbf{x}^{\perp}$ is the one dimensional Lebesgue measure along the line $\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \cdot \vec{\alpha} = s\}$ and $\vec{\alpha} = \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \in \mathbb{R}^2$.

Discrete Tomographic Data

In the pixel-based model, the line integral is discretized using the standard pencil-beam model. We use the pixel-based Matlab routine `radon.m` for simulating parallel-beam tomographic data.

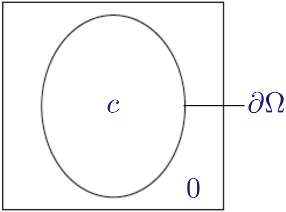
The measurement,

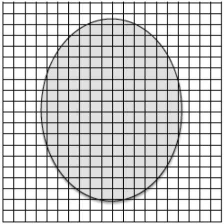
$$m_i = \int_{L_i} f(x,y)ds \approx \Sigma_{j=1}^{n} a_{ij} f_j,$$

where a_{ij} is the distance that L_i travels in the j*th* pixel.

NURBS-based Tomographic Model

The line integral is discretized by moving to pixel-based model using an operator $\mathcal{B} : \mathbb{R}^{2n+1} \to \mathbb{R}^{\mathsf{N}\times\mathsf{N}}$.





$$\mathcal{B}(v) = \begin{cases} c, & \text{if the pixel center is inside the NURBS curve,} \\ 0, & \text{if the pixel center is outside the NURBS curve,} \end{cases} \tag{2}$$

where $v \in \mathbb{R}^{2n+1}$.

Nonlinear Inverse Problem arises

Let $\mathcal{R} : \mathbb{R}^{\mathsf{N}\times\mathsf{N}} \to \mathbb{R}$ and $f : \mathbb{R}^2 \to \mathbb{R}$.

Consider the indirect measurement $\mathbf{m} = \mathcal{R}f + \varepsilon$, where $\mathbf{m} \in \mathbb{R}^k$ and $f = \mathcal{B}(v)$.

The inverse problem is to find f which depends on v when the observation, $\mathbf{m}$ is given.

Overview

Recast inverse problem as a Bayesian inference

We use probability theory to model our lack of information in the inverse problem. All the variables in the model are considered as random variables.

- Construct a prior density (information prior to the measurement)
- Construct likelihood function (the likelihood of different between the observation and the unknown)
- Explore the posterior probability density (what we know about the unknown given observation)

Recast inverse problem as a Bayesian inference

- Our model is $\mathbf{m} = \mathcal{R}(\mathcal{B}(v)) + \varepsilon$.

  Let $\varepsilon \sim \mathcal{N}(0, \sigma^2)$, so then

  $$(\mathbf{m} - \mathcal{R}(\mathcal{B}(v)) \sim \mathcal{N}(0, \sigma^2)$$

- Model of the measurement process:

  $$\pi(\mathbf{m} \,|\, v) = C \exp(-\frac{1}{2\sigma^2} \|\mathcal{R}(\mathcal{B}(v) - \mathbf{m}\|_2^2),$$

  a *likelihood* function.

Construct a priori information

Construct a *priori* information in a quantitative form:

Let $v \sim \mathcal{N}(\tilde{v}, \sigma_2^2)$, so then

$$\pi(v) = \exp(-\frac{1}{2\sigma_2^2}\|v - \tilde{v}\|_2^2), \tag{3}$$

where

$$v = \begin{bmatrix} r_1 \\ \theta_1 \\ . \\ . \\ . \\ \theta_n \\ r_n \\ c \end{bmatrix}, \tilde{v} = \mathbf{V} \in \mathbb{R}^{2n+1}.$$

Construct a posterior distribution

The solution of the inverse problem is the posterior probability distribution:

$$\pi(v \,|\, \mathbf{m}) = \frac{\pi(v)\pi(\mathbf{m} \,|\, v)}{\pi(\mathbf{m})}$$

or

$$\pi(v \,|\, \mathbf{m}) \sim \pi(v)\pi(\mathbf{m} \,|\, v).$$

Overview

Monte Carlo Integration

Consider integral

$$E[g(x)] = \int g(x)\pi(x)dx,$$

where $\pi(x)$ is a probability density and $g \in L^1(\mathbb{R}^n)$.

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where $\pi(x)$ is a probability density, and $g \in L^1(\mathbb{R}^n)$.
In traditional Gaussian quadratures:

$$\int g(x)\pi(x)dx \approx \Sigma_i^K \omega^i g(x^i),$$

a weighted sum of function values at specified points within the domain of integration, where ω^i are the weights and $x^i, i = 1, ..., K$ are the grid points.

Monte Carlo Integration

The Gaussian quadratures is infeasible in high dimensions. It requires K^n integrations points, so then it needs a good knowledge of $\pi(x)$.

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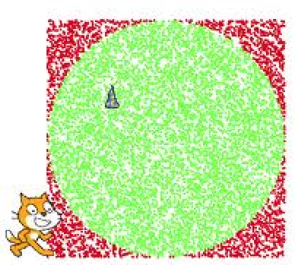
The idea of Monte Carlo is the grid points x^i are generated randomly, choose x^i to be i.i.d. samples of $\pi(x)$.

The *law of large numbers* :

$$\lim \frac{1}{K}\Sigma_i^K g(x^i) = E[g(x)] = \int g(x)\pi(x)dx.$$

Markov Chain Monte Carlo



Monte Carlo approximates expectations with a sample average:

$E(\mathbf{p}) \approx \frac{1}{n}\sum_{i=1}^{n} p_i,$

p_i are i.i.d..

Markov chain Monte Carlo methods involve a Markov process in which a sequence of state p_i is generated.

Each sample p_i has a probability distribution that depend on the previous state p_{i-1}.

Metropolis Hastings

Our model is $\mathbf{m} = \mathcal{R}(\mathcal{B}(v)) + \varepsilon$.

1. Set $l = 0$ and initialize $v^{(0)}$.
2. Draw a random integer k from 1 to number of control points.
3. Set $v := v^k + \epsilon_k$.
4. If $\pi(v|\mathbf{m}) \geq \pi(v^{(l)}|\mathbf{m})$ then set $v^{(l+1)} := v$.
5. Draw a random number s from uniform distribution on $[0, 1]$. If $s \leq \frac{\pi(v|\mathbf{m})}{\pi(v^l|\mathbf{m})}$ then set $v^{(l+1)} = v$, else set $v^{(l+1)} := v^{(l)}$.
6. $l = L$ then stop; else set $l := l + 1$ and go to 2^{nd} step.

Conditional Mean Estimate

The CM (Conditional Mean) estimate is defined by

$$v^{\mathsf{CM}} = \int_{\mathbb{R}^n} \mathbf{v}\pi(\mathbf{v} \,|\, m) d\mathbf{v} = E(\mathbf{v})$$

where $\mathbf{v} = \{v^{(l)}\}_{l=1}^{L}$.

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Using MCMC:

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Then, we recover

$$f^{\mathsf{CM}} = \mathcal{B}(v^{\mathsf{CM}}).$$

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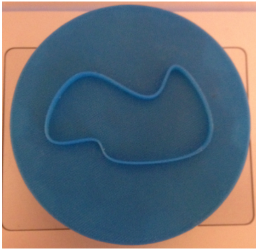
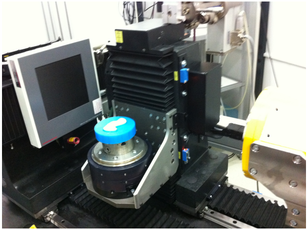
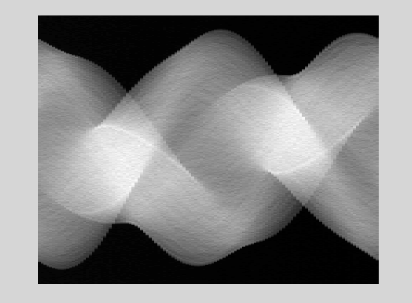
CT Data



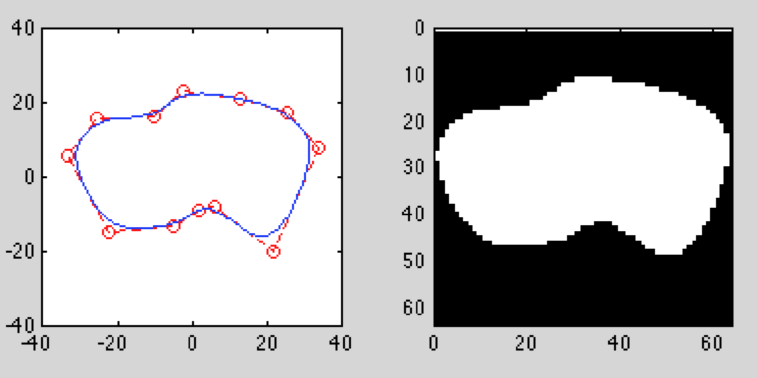




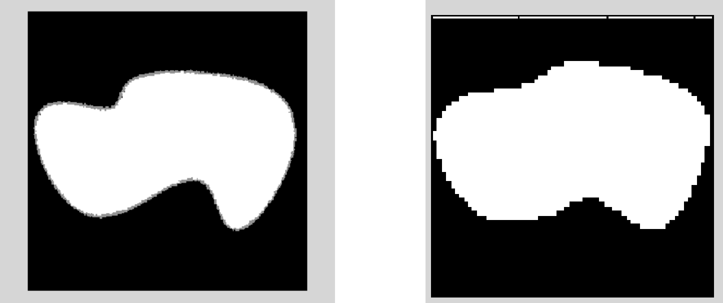
Setting up

Recover 12 control points $\mathbf{p}$ and attenuation c using Metropolis Hasting algorithm with 8 angles.

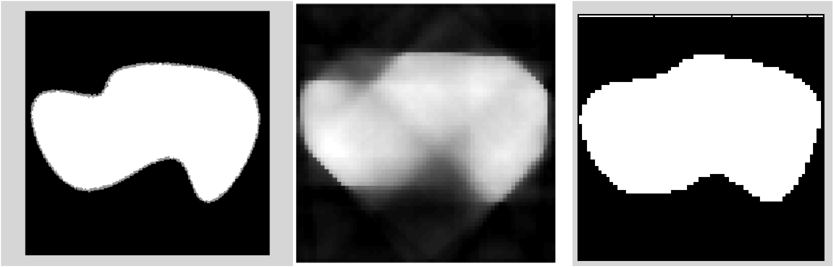
NURBS-MCMC reconstruction



Original and NURBS-MCMC reconstruction



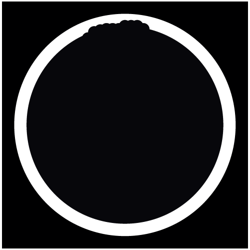
Original, TV and NURBS-MCMC reconstruction



Overview

Courtesy: dammgoodwater.com

Consider homogeneous simple corrosion pipe and set the attenuation is 1 for the pipe and $\frac{1}{30}$ inside the pipe.



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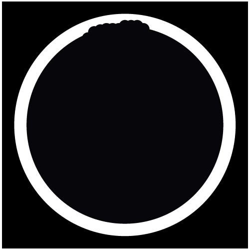
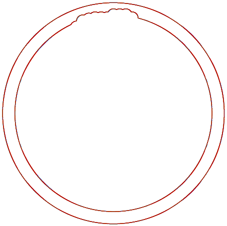
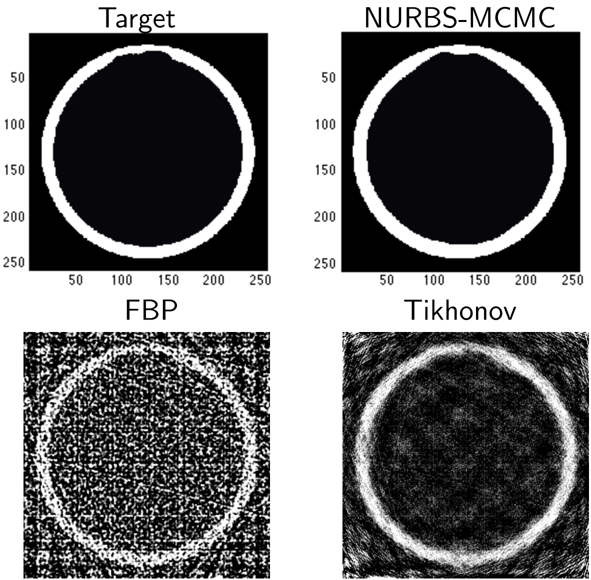




- Recovering 20 control points and the attenuation value where $N = 1000000$ and 12 angles.


Target
NURBS-MCMC
50
100
150
200
250
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- We have demonstrated that NURBS curves combining with MCMC can be used in computational inversion tomography.
- The result is automatically in CAD format (the building blocks of CAD modelling).
- The potential drawback MCMC computation is heavy (expensive) but it can be handle using parallel computing.

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THANK YOU FOR YOUR ATTENTION

Overview

Non Uniform Rational B-Splines (NURBS)

Parametric representation of a curve and surface.

Curve

$$\mathbf{S} : [0, 1] \rightarrow \mathbb{R}^2.$$

They are basically piecewise polynomial functions.

Non Uniform Rational B-Splines (NURBS)

The general form of a NURBS curve is:

$$\mathbf{S}(t) = \frac{\sum_{i=0}^{n} \mathbf{P}_i N_{i,p}(t)\omega_i}{\sum_{i=0}^{n} N_{i,p}(t)\omega_i} = \sum_{i=0}^{n} \mathbf{P}_i R_{i,p}(t),$$

where $N_{i,p}(t)$ are B-splines basis function, $\mathbf{P}_i$ are the control points, ω_i are the weights, and

$$R_{i,p}(t) = \frac{\omega_i N_{i,p}(t)}{\sum_{i=0}^{n} \omega_i N_{i,p}(t)},$$

are the rational B-splines basis function. The $\omega_i \geq 0$ for all values of i.

Important parts in NURBS

- Control Point ($\mathbf{P}_i$)
  A set of points by which the **positions** can determine the NURBS curves.
- Knots (t)
  Defines **how much information should be shared** by segments. This vector divides the curve into intervals. The knots are needed to get the curve to settle in the proper space. A knot vector in one dimension is a set of coordinates in the parametric space, written

$$\mathbf{t} = \{t_1, t_2, ..., t_{n+p+1}\},$$

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$$\mathbf{t} = \{t_1, t_2, ..., t_{n+p+1}\},$$

- Basis Function ($N_{i,p}(t)$)
  A function which determines **how strongly control point, $\mathbf{P}_i$ influences the curve** at time t.

$$N_{i,0}(t) = \begin{cases} 1 & \text{if } t_i \leq t < t_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

$$N_{i,p}(t) = \frac{t - t_i}{t_{i+p} - t_i} N_{i,p-1}(t) + \frac{t_{i+p+1} - t}{t_{i+p+1} - t_{i+1}} N_{1+i,p-1}(t).$$

- Order (p)
  A positive whole number plus zero, refers to **the highest exponent in the polynomial basis function** used for NURBS. $p = 0, 1, 2, 3, etc.$, refers to constant, linear, quadratic, cubic, etc., piecewise polynomials, respectively.

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Knots

Example of $uniform$ knot vector:
$[0 \quad 0.25 \quad 0.5 \quad 0.75 \quad 1.0]$ Some examples of $open$ $uniform$ knot vector :

$$
\begin{aligned}
p = 2, &\qquad [0 \quad 0 \quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{3}{4} \quad 1 \quad 1] \\
p = 3, &\qquad [0 \quad 0 \quad 0 \quad \frac{1}{3} \quad \frac{2}{3} \quad 1 \quad 1 \quad 1] \\
p = 4, &\qquad [0 \quad 0 \quad 0 \quad 0 \quad \frac{1}{2} \quad 1 \quad 1 \quad 1 \quad 1]
\end{aligned}
$$

Knots

Formally, an *open uniform* knot vector is given by

$$\begin{aligned} t_i &= 0, & 0 \le i \le p \\ t_i &= i - p, & p+1 \le i \le n+1 \\ t_i &= n - p + 2, & n+2 \le i \le n+p+1 \end{aligned}$$

Non uniform knot vectors may have either spaced and/or multiple internal knot values. Here are the examples

$$[0 \quad 0 \quad 0.28 \quad 0.5 \quad 0.72 \quad 1]$$

Closed NURBS Curve

- Set the same control point in the ends by using *open uniform* knot vector.
- Repeat the $p-1$ control points by using *periodic uniform* knot vector.