Practical X-ray tomography by total variation reconstruction

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INTRODUCTION

The aim of this work is to practice sparse-angle $2D$ tomography. We consider tomographic data on a walnut taken from 20 angles with 9 degree angular steps, and our goal is to infer the attenuation of different parts of the walnut from the measurements. As a result we get a discrete ill-posed inverse problem.

We build a discretized measurement model where the attenuation of the object is modeled by a 778×778 pixel image $\mathbf{f}$. The attenuation is assumed to be non-negative and constant in each pixel. The pixels are numbered from 1 to $N = 778^2$ and the attenuation is stored as $\mathbf{f} \in \mathbb{R}^N$. The measurements are taken by fan beam geometry but in the calculations we assume parallel beam geometry. The number of measurement directions is $J = 20$ and the number of measurements per direction is 1105, and so the measurement data is collected in $\mathbf{m} \in \mathbb{R}^k$, $k = 22100$.

A measurement m_i gives the line integral of the attenuation $\mathbf{f}$ over the line L_i and is approximated by the sum

$$m_i = \sum_{j=1}^{N} a_{ij} f_j,$$

where a_{ij} is the length of the intersection of L_i and the jth pixel. The lengths a_{ij} comprise the matrix $A \in \mathbb{R}^{k \times N}$. We thus get a discrete measurement model

$$A\mathbf{f} = \mathbf{m} \qquad (1)$$

where $\mathbf{f}$ is to be determined.

Equation (1) is ill-posed, and naïve least squares inversion is highly susceptible to measurement noise. The inevitable presence of noise in practical measurement forces us to use regularization instead of simply looking for a least-squares solution of problem (1) (see [MS]).

METHODS AND MATERIALS

As a reconstruction tool we use total variation (TV) regularization. In contrast to the classical filtered back-projection (FBP), total variation regularization is well-suited to sparse angle tomography. It allows sharp edges in the reconstruction and simultaneously removes noise efficiently by promoting sparsity of the gradient of the attenuation. The idea is to strike a balance between minimizing the discrepancy $\|A\mathbf{f} - \mathbf{m}\|^2$ and minimizing the total variation of $\mathbf{f}$.

Total variation regularization was introduced by Rudin, Osher and Fatemi in [ROF]. For more on total variation regularization in tomographic imaging see *e.g.* [HHHKNS] and [MS].

In order to discretize total variation regularization recall that in MATLAB's enumeration of pixels f_{j+n} is the right neighbor and f_{j+1} the downward neighbor of f_j. We form the horizontal difference operator L_H defined by $(L_H\mathbf{f})_j := f_{j+n} - f_j$ and the vertical difference operator L_V given by $(L_V\mathbf{f})_j := f_{j+1} - f_j$. The functional to be minimized is $G(\mathbf{f}) := \|A\mathbf{f} - \mathbf{m}\|_2^2 + \alpha(\sum_{j=1}^N |(L_H\mathbf{f})_j| + \sum_{j=1}^N |(L_V\mathbf{f})_j|)$, where $\alpha > 0$ is a suitable regularization parameter.

The dimension of the minimization problem is so large that iterative methods are required in order to find an approximate minimizer in reasonable computation time. We use a gradient-based optimization method, and that requires us to replace $|t|$ by $|t|_\beta := \sqrt{t^2 + \beta}$, $\beta > 0$, in the penalty term of G since $|t|$ is not differentiable at zero. The objective functional to be minimized thus becomes

$$G_\beta(\mathbf{f}) := \|A\mathbf{f} - \mathbf{m}\|_2^2 + \alpha \left(\sum_{j=1}^{N} |(L_H\mathbf{f})_j|_\beta + \sum_{j=1}^{N} |(L_V\mathbf{f})_j|_\beta \right).$$

In this work we choose the parameter value $\beta = 0.000001$.

In the minimization of G_β we use the Barzilai-Borwein (BB) method introduced in [BB]. (To be quite precise, we use the projected Barzilai-Borwein (PBB) method since we enforce the constraint $\mathbf{f} \geq 0$ at each step of the iteration.) The BB method is a modification of the classical steepest descent (SD) method; the step direction is the same as in SD but the step length is chosen without having to perform a computationally costly line search. The BB method is well-suited to the study of optimization problems of a very large scale; it requires few values to be stored and few computations. Furthermore, it converges much faster than the SD method. For more on the BB method see *e.g.* [Fle] and the references contained therein. The PBB method is applied to total variation reconstruction in [HHHKNS].

We next describe the BB method in the case of the optimization problem at hand. When ℓ iteration steps of $\mathbf{f}$ and $G_\beta(\mathbf{f})$ have been computed, in quasi-Newton methods one sets

$$\mathbf{f}^{(\ell+1)} = \mathbf{f}^{(\ell)} - (B_\ell)^{-1} \nabla G_\beta(\mathbf{f}^{(\ell)}) \qquad (2)$$

where $B_\ell \in \mathbb{R}^{N \times N}$ is an approximation of the Hessian $\nabla^2 G_\beta(\mathbf{f}^{(\ell)})$. Usually B_l is chosen to satisfy the secant equation $B_\ell y_\ell = g_\ell$, where

$$\begin{aligned} y_\ell &:= \mathbf{f}^{(\ell)} - \mathbf{f}^{(\ell-1)}, \\ g_\ell &:= \nabla G_\beta(\mathbf{f}^{(\ell)}) - \nabla G_\beta(\mathbf{f}^{(\ell-1)}). \end{aligned}$$

Barzilai and Borwein chose in (2) the first-order approximation $B_\ell = \alpha_\ell I$, where $\alpha_\ell \in \mathbb{R}$ gives the least-squares solution of the secant equation $(\alpha_\ell I) y_\ell = g_\ell$.

Once α_ℓ is computed, the BB method obtains the form

$$\mathbf{f}^{(\ell+1)} = \mathbf{f}^{(\ell)} - \delta_\ell \nabla G_\beta(\mathbf{f}^{(\ell)}),$$

where the step length δ_ℓ is given by the formula

$$\delta_\ell = \frac{y_\ell^T y_\ell}{y_\ell^T g_\ell}.$$

We choose the first step length by a steepest descent type line search within the logarithmic scale from 10^{-6} to 10^{-1}. We stress that *the BB method is non-monotone*, that is, the value of G_β does not necessarily decrease at every iteration!

We select the regularization parameter $\alpha > 0$ by using the S-curve method of Kolehmainen, Lassas, Niinimäki and Siltanen (see [MS, p. 89]). We first photograph a sawed walnut and calculate the amount of essentially nonzero coefficients in the Fourier transform of the attenuation; this is used as an estimate of the number of essentially nonzero coefficients for the walnut we use in the measurement.

We then use parallel computation to form the total variation reconstruction $\mathbf{f}$ for different values of parameter α which reside on a logarithmic scale from 10^{-6} to 10. By spline interpolation we choose α such that the number of essentially nonzero coefficients of the Fourier transform of $\mathbf{f}$ is closest to the estimate we obtained from the sawed walnut.

The total variation reconstruction is obtained by using the parameter α picked by the S-curve method and performing 200 iterations in the BB method. We calculate the relative reconstruction error by comparing the reconstruction to the "ground truth" which is obtained by forming a filtered back-projection reconstruction of the walnut with 180 measurement angles.

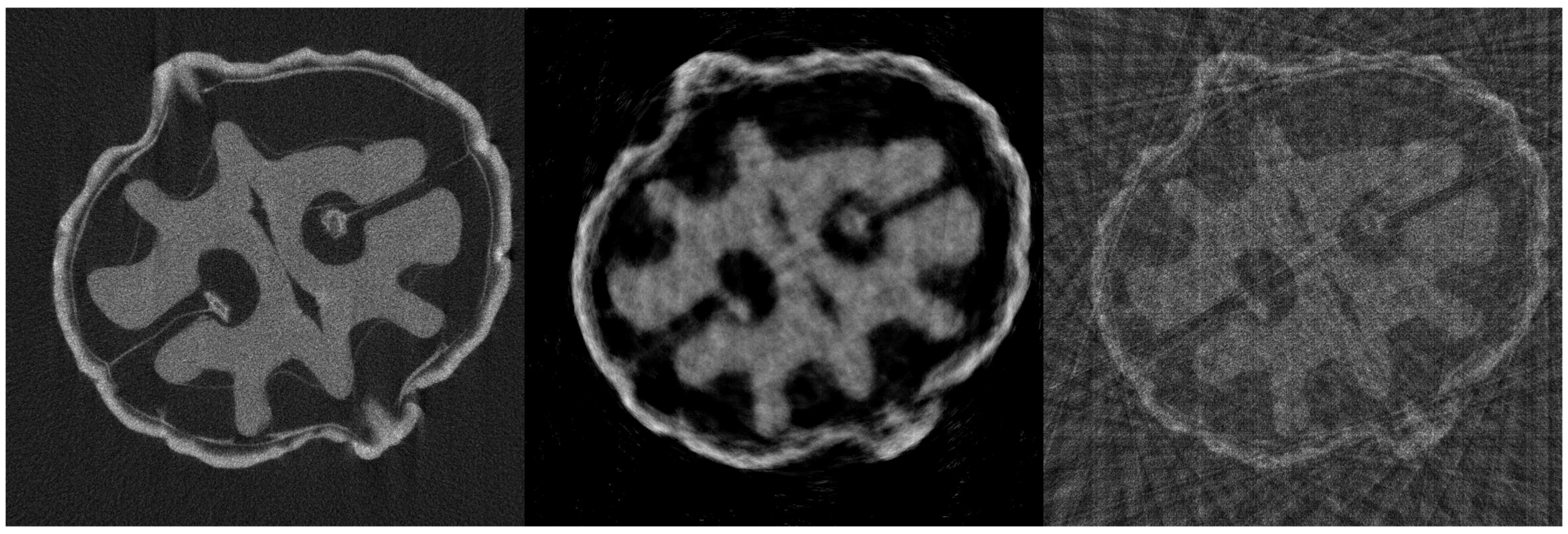

Figure 1: Left: Ground truth. Middle: TV reconstruction with 20 measurement angles, relative square norm error 43%. Right: FBP reconstruction with 20 measurement angles, relative square norm error 122%.

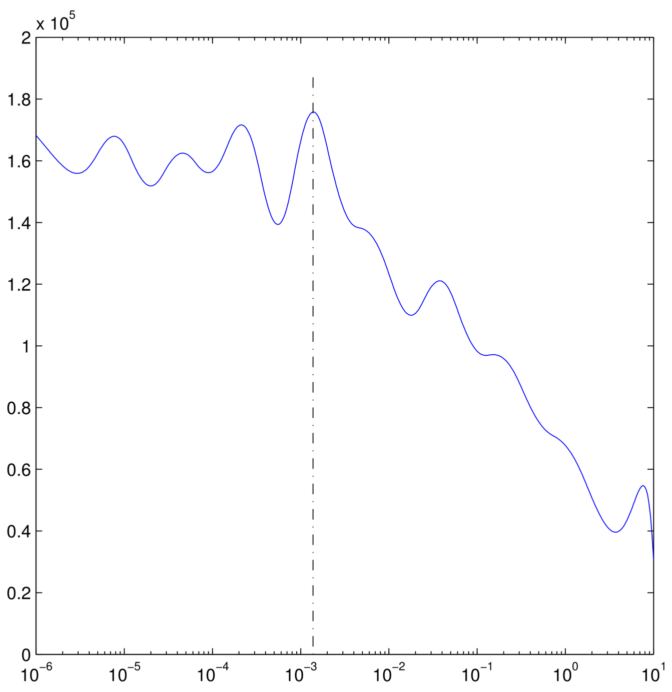

Figure 2: Plot of the S-curve. The vertical dashed line shows the interpolated value $\alpha = 0.0014$ that is closest to the estimate from the sawed walnut.

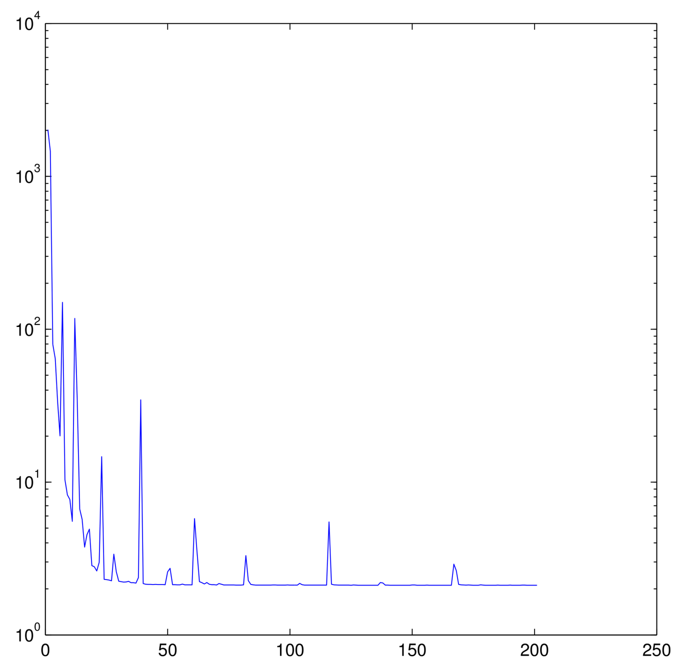

Figure 3: Evolution of the objective functional G_β during the 200-step Barzilai-Borwein iteration process.

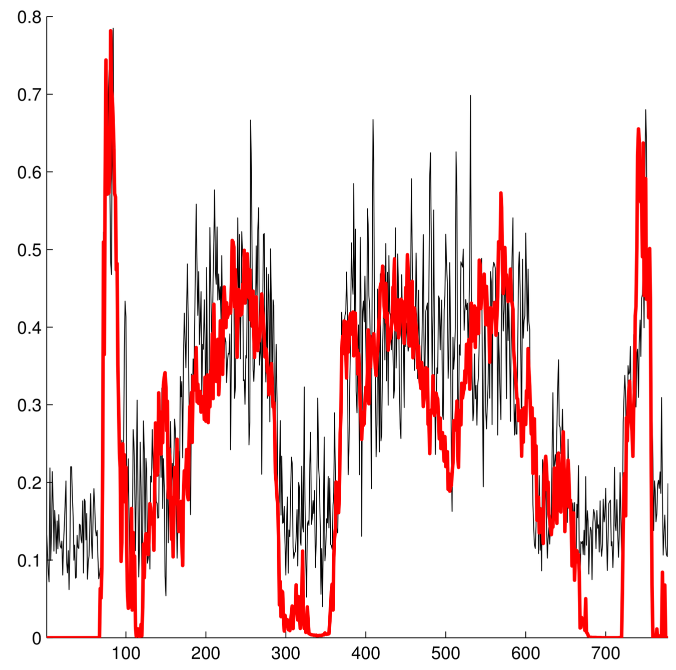

Figure 4: The normalized attenuation values of the ground truth (black) and the TV reconstruction (red) at a single row in the middle of the pixel image.

RESULTS

The results of the S-curve method are shown in Figure 2. The value picked by the method is $\alpha = 0.0014$. A plot of the evolution of the objective functional G_β is shown in Figure 3. Note that a rather steady level of G_β is achieved already in around 30 to 40 iterations.

The TV reconstruction is shown in the middle of Figure 1; the FBP reconstruction with 20 measurement angles is also reproduced for comparison. The relative square norm error of the TV reconstruction is 43% and that of the FBP reconstruction is 122%. Figure 4 illustrates the variation of the attenuation within the walnut.

DISCUSSION

The results support the contention that total variation regularization is an efficient method of $2D$ tomographic imaging when few measurement angles are available. The great speed of convergence of the Barzilai-Borwein method was, however, a surprise to the authors.

This work could be extended to many directions. One possible further project would be to monitor the evolution of the square norm error when the number of measurement angles is decreased.

References

[BB] Barzilai J and Borwein J M, *Two-Point Step Size Gradient Methods*, IMA Journal of Numerical Analysis 8 (1988), 141-148.

[MS] Mueller J L and Siltanen S, *Linear and Nonlinear Inverse Problems with Practical Applications*, SIAM, Philadelphia, 2012.

[Fle] Fletcher R, *On the Barzilai-Borwein method*, Optimization and Control with Applications 96 (2005), 235-256.

[HHHKNS] Hämäläinen K, Harhanen L, Hauptmann A, Kallonen A, Niemi E and Siltanen S, *Total variation regularization for large-scale X-ray tomography*, International Journal of Tomography and Simulation 25 (2014), 1-25.

[ROD] Rudin L I, Osher S and Fatemi E, *Nonlinear total variation based noise removal algorithms*, Physica D 60 (1992), 259-268.