Inverse problems course — Exercise 2 (January 27-29, 2015)
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Related book sections (Mueller & Siltanen 2012): 2.1.1, 2.1.2 and 3.5.

Theoretical exercises:

T1. Assume that the $n \times n$ matrix $U : \mathbb{R}^n \to \mathbb{R}^n$ is orthogonal: $UU^T = I = U^T U$.

(a) Show that $\|U^T y\| = \|y\|$ for any $y \in \mathbb{R}^n$.

(b) Let $x, y \in \mathbb{R}^n$. Show that the angle between the vectors x and y is the same than the angle between the vectors Ux and Uy.

T2. Let A be a real-valued $n \times n$ matrix.

(a) Show that the matrix $A^T A$ is symmetric.

(b) Show that if λ is an eigenvalue of $A^T A$, then $\lambda \geq 0$.

T3. Let A be a real-valued $n \times n$ matrix. Recall from basic linear algebra that a symmetric matrix can be diagonalized and its eigenvectors can be chosen to be orthonormal. Denote the eigenvalues of $A^T A$ by

$$d_1^2 \geq d_2^2 \geq \cdots \geq d_r^2 > d_{r+1}^2 = d_{r+1}^2 = \cdots = d_n^2 = 0,$$

and the corresponding orthonormal eigenvectors by $V^{(1)}, V^{(2)}, \ldots, V^{(n)}$. Insert the eigenvectors as columns to a matrix called V. Also, write $V = [V_1\, V_2]$ with

$$V_1 = [V^{(1)}\, V^{(2)} \cdots V^{(r)}], \qquad V_2 = [V^{(r+1)}\, V^{(r+2)} \cdots V^{(n)}].$$

Then

$$V^T A^T A V = \begin{bmatrix} \Sigma^2 & 0 \\ 0 & 0 \end{bmatrix},$$

where the $r \times r$ matrix Σ is defined by $\Sigma^2 = \operatorname{diag}(d_1^2, \ldots, d_r^2)$. Here $V_1^T A^T A V_1 = \Sigma^2$. **Show that** $AV_2 = 0$. Now define a $n \times r$ matrix U_1 by $U_1 = AV_1\Sigma^{-1}$. **Show that** $U_1^T U_1 = I$. Therefore the columns of U_1 are orthonormal. **Show that** we can define an orthonormal $n \times n$ matrix in the form $U = [U_1\, U_2]$. **Finally, derive the SVD by showing that**

$$U^T A V = \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix}.$$

Hint: use the block forms of the matrices.

Matlab exercises:

M1. Let us study quantitative comparisons of reconstructions. Consider given a "truth" vector $f \in \mathbb{R}^n$ and a "reconstruction" or "approximation" vector $g \in \mathbb{R}^n$. Define *relative error* by the formula

$$\frac{\|f - g\|}{\|f\|} \cdot 100\%.$$

(a) Construct crime-free convolution data with and without added noise as is done in the routine `deconv3_naive_comp.m`. Compute the relative error of the noisy data compared to the non-noisy data. How does the percentage compare to the parameter `noiselevel`? Does this relationship depend on the dimension `n`?

(b) Use truncated singular value decomposition (`deconv5_truncSVD_comp.m`) to compute reconstructions of a target from noisy crime-free data. Which number r_α of singular values gives the smallest relative error in the reconstruction? How does the optimal value of r_α change if you increase the noise level?

M2. Create your own target function, compute noisy crime-free convolution data of it, and reconstruct it using truncated singular value decomposition and file `deconv5_truncSVD_comp.m`. Show the reconstruction (but not the original target) to some other student. How well can he or she recover the true function from the reconstruction?