Martingale Predictable Representation Property, hedging, and binomial tree

Dario Gasbarra

17. maaliskuuta 2014

$$V_t = \beta_t B_t + \gamma_t S_t = \beta_{t+1} B_t + \gamma_{t+1} S_t \qquad \text{(self-financing condition)}$$

$$\widetilde{S}_t = S_t / B_t, \quad \widetilde{V}_t = V_t / B_t, \quad \widetilde{V}_t = V_t / B_t, \quad \widetilde{G} = G / B_T$$

$$\widetilde{V}_t = E_Q(\widetilde{G} | \mathcal{F}_t) = E_Q(\widetilde{G} | \mathcal{F}_0) + \sum_{u=1}^{T} \gamma_u d\widetilde{S}_u$$

($\widetilde{S}_t$ has the PRP in the filtration $\mathbb{F}$)

$$\widetilde{V}_T = \widetilde{G}$$

By Abel discrete integration by parts formula

$$V_t = \widetilde{V}_t B_t = \widetilde{V}_0 B_0 + \sum_{u=1}^{t} B_u \Delta \widetilde{V}_u + \sum_{u=1}^{t} \widetilde{V}_{u-1} \Delta B_u$$

$$= c_0(G) + \sum_{u=1}^{t} B_u \gamma_u \Delta \widetilde{S}_u + \sum_{u=1}^{t} \frac{V_{u-1}}{B_{u-1}} \Delta B_u$$

where

$$\Delta \widetilde{S}_u = \frac{1}{B_u} \Delta S_u - \frac{S_{u-1}}{B_u B_{u-1}} \Delta B_u$$

and we get

$$V_t = c_0(G) + \sum_{u=1}^{t} \gamma_u \Delta S_u + \sum_{u=1}^{t} \frac{\left(V_{u-1} - \gamma_u S_{u-1}\right)}{B_{u-1}} \Delta B_u$$

$$= c_0(G) + \sum_{u=1}^{t} \gamma_u \Delta S_u + \sum_{u=1}^{t} \eta_u \Delta B_u$$

$$G = V_T = c_0(G) + \sum_{u=1}^{T} \gamma_u \Delta S_u + \sum_{u=1}^{T} \eta_u \Delta B_u$$

at time $t = T$

Note that it is not necessary to assume that the numeraire B_t is $\mathbb{F}$-predictable.

Consider the finite probability space $(\Omega, \mathcal{F}, P)$ where $\Omega = \{0,1\}^T$, with $T < \infty$, and $\mathcal{F} = 2^\Omega$, the finite collection of all possible subset, and probability measure satisfies $P(\{\omega\}) > 0$ for all $\omega \in \Omega$.
here $\mathcal{F}_0 = \{\Omega, \emptyset\}$ is the trivial σ-algebra.
An history is a vector $\omega = (\omega_1, \dots, \omega_T) \in \Omega$ and denote $\omega^t = (\omega_1, \dots, \omega_t)$ for $t \leq T$.

Consider a market with a bank account B_t and a stock price S_t, $t = 0, 1, \ldots, T$, adapted to the filtration $\mathbb{F}$ with $\mathcal{F}_t = \sigma(\omega_s, s \leq t)$, $\mathcal{F}_0 = \{\Omega, \emptyset\}$
We assume that there are $\{\mathcal{F}_t\}$-**predictable** processes $U_t(\omega) > R_t(\omega) > D_t(\omega) > -1$. $B_0 > 0$ and $S_0 > 0$ are determistic values, and we let

$$B_t = B_0 \prod_{s=1}^{t} (1 + R_t),$$

$$S_t = S_0 \prod_{s=1}^{t} (1 + D_t + \omega_t (U_t - D_t))$$

Suppose that $G(\omega)$ is a $\mathcal{F}_t$-measurable contingent claim, and we want to find a self-financing hedging strategy (β_t, γ_t) satisfying

$$V_t = \beta_t B_t + \gamma_t S_t = \beta_{t+1} B_t + \gamma_{t+1} S_t \ .$$

Let $\bar{G}(\omega) = G(\omega)/B_T(\omega)$ the discounted contingent claim. We show first that there is an unique probability measure Q such that $Q \sim P$ and the discounted process $\bar{S}_t := (S_t/B_t)$ is a Q-martingale.
Once we have shown that Q is the unique martingale measure for $(\bar{S}_t)$ in the filtration $\mathbb{F}$, it follows that every $(Q, \mathbb{F})$ martingale (N_t) has the representation as

$$N_t = N_0 + \sum_{u=1}^{t} H_u \Delta \bar{S}_u$$

where (H_t) is a $\mathbb{F}$-predictable process.

In particular we can take $N_t = E_Q(\bar{G}|\mathcal{F}_t)$, and obtain when $t = T$

$$\bar{G}(\omega) = \frac{G(\omega)}{B_T(\omega)} = E_Q(\bar{G}|\mathcal{F}_T) = E_Q(\bar{G}) + \sum_{t=1}^{T} \gamma_t \Delta \bar{S}_t$$

where (γ_t) is a $\mathbb{F}$-predictable process.
This gives the unique price $c(G) = E_Q(\bar{G})B_0$ and the hedging strategy for the contingent claim G.

Lets' first compute the martingale measure Q.

$$\Delta\bar{S}_t = \left(\frac{S_t}{B_t} - \frac{S_{t-1}}{B_{t-1}}\right) =$$
$$\frac{S_{t-1}}{B_{t-1}}\left(\frac{(1 + D_t + (U_t - D_t)\omega_t)}{(1 + R_t)} - 1\right) =$$
$$\frac{S_{t-1}}{B_{t-1}(1 + R_t)}((U_t - D_t)\omega_t - (D_t - R_t))$$

Taking conditional expectation with respect to a measure Q, and imposing the martingale property

$$E_Q(\Delta \bar{S}_t | \mathcal{F}_{t-1}) =$$
$$\frac{S_{t-1}}{B_{t-1}(1+R_t)}\big((U_t - D_t)E_Q(\omega_t|\mathcal{F}_{t-1}) - (D_t - R_t)\big) = 0$$

This implies that Q is a martingale measure for $(\bar{S}_t)$ if and only if

$$q_t(\omega^{t-1}) := E_Q(\omega_t|\mathcal{F}_{t-1}) = \frac{(R_t - D_t)}{(U_t - D_t)},$$

where $q_t(\omega^{t-1}) \in (0,1)$ is a probability since we have assumed that $D_t < R_t < U_t$, P a.s, and it is uniquely determined.

We define globally the unique risk-neutral measure Q as follows:

$$Q(\omega) = \prod_{t=1}^{T} q_t(\omega^{t-1})^{\omega_t}(1 - q_t(\omega^{t-1}))^{1-\omega_t}$$

and note that $Q(\{\omega\}) > 0$ for all $\omega \in \Omega$, therefore $Q \sim P$. We define the basic Q-martingale

$$M_t = \sum_{s=1}^{t} (\omega_s - q_s(\omega^{(s-1)}))$$

We write

$$\Delta \bar{S}_t = \frac{S_{t-1}}{B_{t-1}(1+R_t)}(U_t - D_t)(\omega_t - q_t(\omega^{(t-1)}))$$
$$= \frac{S_{t-1}}{B_{t-1}(1+R_t)}(U_t - D_t)\Delta M_t$$

and we can represent ΔM_t in terms of $\Delta \bar{S}_t$:

$$\Delta M_t = \frac{B_{t-1}(1+R_t)}{S_{t-1}(U_t - D_t)}\Delta \bar{S}_t$$

Next we show how to use the martingale representation to compute the hedging strategy for the contingent claim G.

Definition

If $X(\omega)$ is a $\mathcal{F}_T$-measurable random variable, we define its discrete Malliavin derivative or stochastic gradient at time t w.r.t ω_t as

$$\begin{aligned} \nabla_t X(\omega) \quad &:= \quad X(\omega_1, \ldots, \omega_{t-1}, 1, \omega_{t+1}, \ldots, \omega_T) \\ & \qquad -X(\omega_1, \ldots, \omega_{t-1}, 0, \omega_{t+1}, \ldots, \omega_T) \end{aligned}$$

for $1 \leq t \leq T$.

Note that in general $\nabla_t X(\omega)$ is not $\mathcal{F}_t$ measurable unless the r.v. $X(\omega) = X(\omega^t)$ is $\mathcal{F}_t$-measurable. In such case $\nabla_t X(\omega)$ is also $\mathcal{F}_{t-1}$-measurable.

In particular the following quantities are $\mathcal{F}_{T-1}$-measurable.

$$\nabla_T G(\omega^{T-1}) = (G(\omega^{T-1}, 1) + G(\omega^{T-1}, 0)) \quad ,$$

$$\nabla_T \bar{G}(\omega^{T-1}) = (\bar{G}(\omega^{T-1}, 1) + \bar{G}(\omega^{T-1}, 0))$$

$$= \frac{1}{B_T(\omega)}(G(\omega^{T-1}, 1) + G(\omega^{T-1}, 0))$$

$$= \frac{\nabla_T G(\omega^{T-1})}{B_T(\omega)}$$

since $B_T(\omega)$ is $\mathcal{F}_{T-1}$-measurable, and

$$\nabla_T S_T(\omega^{T-1}) = (S_T(\omega^{T-1}, 1) + S_T(\omega^{T-1}, 0))$$

$$= S_{T-1}(U_T(\omega^{T-1}) - D_T(\omega^{T-1})) \, .$$

$$\nabla_T \bar{S}_T(\omega^{T-1}) = \frac{1}{B_T} \nabla_T \bar{S}_T(\omega^{T-1})$$

Note also that

$$\Delta \bar{S}_T = (\bar{S}_T - \bar{S}_{T-1}) = \frac{S_{T-1}}{B_T}(U_T - D_T)(\omega_T - q_T)$$
$$= \nabla_T \bar{S}_T(\omega_T - q_T)$$

so that we can write

$$\Delta M_T = (\omega_T - q_T(\omega^{T-1})) = \frac{1}{\nabla_T \bar{S}_T}\Delta \bar{S}_T = \frac{B_T}{\nabla_T S_T}\Delta \bar{S}_T$$

We have

$$\begin{aligned}
&\bar{G}(\omega) = \bar{G}(\omega^{T-1}, \omega_T) = \\
&\bar{G}(\omega^{T-1}, 0) + (\bar{G}(\omega^{T-1}, 1) - \bar{G}(\omega^{T-1}, 0))\omega_T = \\
&\bar{G}(\omega^{T-1}, 0) + \nabla_T \bar{G}(\omega^{T-1})\omega_T = \\
&\bar{G}(\omega^{T-1}, 0) + \nabla_T \bar{G}(\omega^{T-1}) q_T + \nabla_T \bar{G}(\omega^{T-1})(\omega_T - q_T) = \\
&E_Q(\bar{G}|\mathcal{F}_{T-1}) + \nabla_T \bar{G}\ \Delta M_T = E_Q(\bar{G}|\mathcal{F}_{T-1}) + \frac{\nabla_T \bar{G}}{\nabla_T S_T} B_T \Delta \bar{S}_T \\
&= E_Q(\bar{G}|\mathcal{F}_{T-1}) + \frac{\nabla_T G}{\nabla_T S_T} \Delta S_T - \frac{\nabla_T G}{\nabla_T S_T} R_T S_{T-1} \\
&= E_Q(\bar{G}|\mathcal{F}_{T-1}) + \frac{\nabla_T G}{\nabla_T S_T} \Delta S_T - \frac{\nabla_T G}{\nabla_T S_T} \frac{S_{T-1}}{B_{T-1}} \Delta B_t
\end{aligned}$$

By investing at time $(T-1)$ the (random) value

$$c_{T-1}(G) = E_Q(\bar{G}|\mathcal{F}_{T-1}(\omega)B_{T-1}(\omega) = \frac{E_Q(G|\mathcal{F}_{T-1})(\omega)}{1+R_T}$$

we replicate the contingent claim G as follows: we buy the amount of stocks

$$\gamma_T = \frac{\nabla_T G}{\nabla_T S_T}$$

at price $\gamma_T S_{T-1}$ (if $\gamma_T < 0$ we short-sell stocks) , if necessary by borrowing from the bank at the predictable interest rate R_T, and buy the amount of

$$\beta_T = \frac{1}{B_{T-1}}\Big(c_{T-1}(G) - \gamma_T S_{T-1}\Big)$$

bonds at price B_{T-1}, so that our capital is

$$V_{T-1} = c_{T-1}(G) = \beta_T B_{T-1} + \gamma_T S_{T-1}$$

At time $(T-1)$ the value of our portfolio is

$$V_{T-1} = \beta_T B_{T-1} + \gamma_T S_{T-1} = c_{T-1}(G)$$

while at time T the value of the portfolio becomes

$$\begin{aligned}
&V_T = \beta_T B_T + \gamma_T S_T = \beta_T B_{T-1}(1+R_T) + \gamma_T S_{T-1} + \gamma_T \Delta S_T \\
&= E_Q(G|\mathcal{F}_{T-1}) - \gamma_T S_{T-1}(1+R_T) + \gamma_T S_{T-1} + \gamma_T \Delta S_T \\
&= E_Q(G|\mathcal{F}_{T-1}) - \gamma_T S_{T-1} R_T + \gamma_T \Delta S_T = \\
&E_Q(G|\mathcal{F}_{T-1}) + \gamma_T (S_T - (1+R_T) S_{T-1}) = \\
&E_Q(G|\mathcal{F}_{T-1}) + B_T \gamma_T \Delta \bar{S}_T = G(\omega)
\end{aligned}$$

Remark The martingale measure Q when it is unique gives a device to compute the price and hedging strategy. In fact the price hedging can be computed without using probability, once we have assumed that all histories $\omega \in \Omega$ have positive probability:

A direct way to compute the hedging without using martingales is to solve at time T the system of equations:

$$G(\omega^{T-1},0) = B_T\beta_T + \gamma_T S_{T-1}(1+D_T)$$
$$G(\omega^{T-1},1) = B_T\beta_T + \gamma_T S_{T-1}(1+U_T)$$

By substracting these two equations we get

$$\gamma_T = \frac{\nabla_T G(\omega^{T-1})}{S_{T-1}(U_T - D_T)}$$

and if the two equations with respective weights $(1 - q_T(\omega^{T-1}))$ corresponding to $\omega_T = 0$ and $q_T(\omega^{T-1})$ corresponding to $\omega_T = 1$ we obtain

$$\beta_T = \frac{1}{B_T}\left(E_Q(G|\mathcal{F}_{T-1}) - \gamma_T E_Q(S_T|\mathcal{F}_{T-1})\right)$$
$$= \frac{1}{B_T}E_Q(G|\mathcal{F}_{T-1}) - \gamma_T\frac{S_{T-1}}{B_{T-1}}$$

combining these toghether we get the price of the contingent claim at time $(T-1)$:

$$c_{T-1}(G) = \beta_T B_{T-1} + \gamma_T S_{T-1} = \frac{1}{1+R_T} E_Q(G|\mathcal{F}_{T-1})$$

The martingale method has the advantage that it gives a probabilistic interpretation to the price of the contingent claim, which can be computed directly as a Q-expectation. The other reason is that the martingale method can be extended to the continuous-time setting.

The price and the hedging strategy in the whole time interval $t = 1, \dots, T$, is then obtained by induction:
Let $c_t(G)$ be the price of the contract G at time $t \leq T$. This is a $\mathcal{F}_t$-measurable contingent claim. This means that are able to hedge the contingent claim G expiring at time T if and only if at time t we own a portfolio of value $c_t(G)$. By repeating the martingale argument or by writing directly the system of equations we find the price of the contract at time $(t-1)$ $c_{t-1}(G)$ and the replicating portfolio $\beta_t(\omega^{t-1}), \gamma_t(\omega^{t-1})$.

The advantage the martingale method is that enables to compute directly price and replicating strategy at all times t by computing Q-expectations.
The predictable representation property of the Q-martingale M gives

Theorem

Discrete Clarck-Ocone formula:

$$E_Q(\bar{G}|\mathcal{F}_t)(\omega) =$$
$$E_Q(\bar{G}) + \sum_{s=1}^{t} \nabla_s E_Q(\bar{G}(\omega)|\mathcal{F}_s)\big(\omega_s - q_s(\omega^{s-1})\big)$$
$$= E_Q(\bar{G}) + \sum_{u=1}^{t} \frac{\nabla_u E_Q(\bar{G}(\omega)|\mathcal{F}_u)}{\nabla_u \bar{S}_u} \Delta \bar{S}_u$$

where by definition $\nabla_t E_Q(\bar{G}(\omega)|\mathcal{F}_t)$ *is* $\mathcal{F}_{t-1}$*-measurable.*

We set

$$\gamma_t = \frac{\nabla_t E_Q(G(\omega)|\mathcal{F}_t)}{\nabla_t S_t}$$

This gives

$$\begin{aligned} V_t &= E_Q(G|\mathcal{F}_t) = E_Q(G|\mathcal{F}_{t-1}) + \gamma_t B_t \Delta \bar{S}_t \\ &= \frac{E_Q(G|\mathcal{F}_{t-1})}{1+R_t} + \gamma_t \Delta S_t + \left(\frac{E_Q(G|\mathcal{F}_{t-1})}{1+R_t} - \gamma_t S_{t-1} \right) \frac{1}{B_{t-1}} \Delta B_t \\ &= V_{t-1} + \gamma_t \Delta S_t + \beta_t \Delta B_t \end{aligned}$$

where

$$\beta_t = \left(\frac{E_Q(G|\mathcal{F}_{t-1})}{1+R_t} - \gamma_t S_{t-1} \right) \frac{1}{B_{t-1}}$$

This means that to obtain a portfolio with value $E_Q(G|\mathcal{F}_t)$ at time t, we need to invest

$$c_{t-1} := E_Q(G|\mathcal{F}_{t-1})/(1 + R_t)$$

at time $(t-1)$. Equivalently, to have $E_Q(G\frac{B_t}{B_T}|\mathcal{F}_t)$ in our portfolio at time t we need to invest the amount

$$E_Q(G\frac{B_{t-1}}{B_T}|\mathcal{F}_{t-1}) \quad \text{at time } (t-1)\ .$$

Inductively , to have $G = E_Q(G|\mathcal{F}_T)$ at time T we have to invest at time $s \leq T$ the amount

$$c_t(G) = E_Q(G\frac{B_t}{B_T}|\mathcal{F}_t)$$

at time t.
The hedging at time $(t-1)$ is given by

$$\gamma_t = \frac{\nabla_t E_Q(G(\omega)\frac{B_t}{B_T}|\mathcal{F}_t)}{\nabla_t S_t} = \frac{\nabla_t c_t(G)}{\nabla_t S_t},$$
$$\beta_t = \left(c_{t-1}(G) - \gamma_t S_{t-1}\right)\frac{1}{B_{t-1}}$$

and we get

$$V_t = c_t(G) = c_0(G) + \sum_{u=1}^{t} (\gamma_u \Delta B_u + \beta_u \Delta B_u)$$

$$V_T = G = c_0(G) + \sum_{u=1}^{T} (\gamma_u \Delta B_u + \beta_u \Delta B_u)$$

When R_t is deterministic, we can take the discounting factors B_t / B_T outside the conditional expectation.
If (D_t, R_t, U_t) are all deterministic, then under the martingale measure Q the random variables ω_t is independent from the past. Then the computation of the hedging strategy may be simplified by using the following formula:

Corollary

If (D_t, R_t, U_t) are deterministic at all $t \leq T$, conditional expectation and gradient commute in Ito-Clarck formula

$$\nabla_t E_Q(G|\mathcal{F}_t) = E_Q(\nabla_t G|\mathcal{F}_t) = E_Q(\nabla_t G|\mathcal{F}_{t-1}) \; ,$$

giving

$$E_Q(G|\mathcal{F}_t)(\omega) = E_Q(G) + \sum_{s=1}^{t} E_Q(\nabla_s G|\mathcal{F}_s)\big(\omega_s - q_s(\omega^{s-1})\big) \; .$$

Proof When $\omega = (\omega_1, \dots, \omega_T)$ we denote $\omega^{t,T}$ the vector $(\omega_t, \dots, \omega_T)$.
Using the independence of the r.v. (ω_t),

$$E_Q(\nabla_t G|\mathcal{F}_t)(\omega_t) = \sum_{\omega^{t+1,T} \in \{0,1\}^{T-t}} \left\{ G(\omega^{t-1}, 1, \omega^{t+1,T}) - G(\omega^{t-1}, 0, \omega^{t+1,T}) \right\} \times Q(\omega^{t+1,T}) = \nabla_t E_Q(G|\mathcal{F}_t)(\omega_t)$$

which is $\mathcal{F}_{t-1}$-measurable.

Example Assume that $R_t = r, U_t = u, D_t = d$ deterministic, with $-1 < d < r < u$. Then $q_t = q = (r - d)/(u - d)$ is constant. We have that

$$S_t = S_0(1 + u)^{N_t}(1 + d)^{t-N_t}$$

where $N_t = \sum_{s=1}^{t} \omega_s$.

Then if $G(\omega) = \varphi(S_T)$ is a plain european option, we compute the price at time $t = 0$ using the distribution Binomial(q, T).

$$V_0 = c_0(G) = B_0 E_Q(\varphi(S_T)/B_T) =$$

$$(1+r)^{-T} \sum_{n=0}^{T} \binom{T}{n} q^n (1-q)^{T-n} \varphi\big(S_0(1+u)^n(1+d)^{T-n}\big) \ .$$

Similarly since the conditional distribution of $(N_T - N_t)$ given $\mathcal{F}_t$ is Binomial$(q, T - t)$, at time t the price of the replicating portfolio is

$$V_t = c_t(G) = B_t E_Q(\varphi(S_T)/B_T|\mathcal{F}_t) =$$
$$(1+r)^{t-T} \sum_{n=0}^{T-t} \binom{T-t}{n} q^n (1-q)^{T-t-n}$$
$$\times \varphi\Big(S_0(1+u)^{N_t+n}(1+d)^{T-N_t-n}\Big) \, .$$

with this amount of money, we invest in γ_{t+1} stocks and invest the rest in the bank account, with

$$\gamma_{t+1} = \frac{\nabla_{t+1} c_{t+1}(G)}{\nabla_{t+1} S_{t+1}} =$$

$$(1+r)^{t+1-T} \frac{E_Q(\nabla_{t+1} G | \mathcal{F}_t)}{S_t(u-d)} =$$

$$(1+r)^{t+1-T} \frac{1}{S_t(u-d)} \sum_{n=0}^{T-t-2} \left\{ \binom{T-t-2}{n} q^n (1-q)^{T-t-2-n} \times \right.$$

$$\times \Big(\varphi\big(S_0(1+u)^{N_t+n+1}(1+d)^{T-N_t-n-2}\big)$$

$$\left. - \varphi\big(S_0(1+u)^{N_t+n}(1+d)^{T-N_t-n-1}\big) \Big) \right\}$$