Introduction to differential forms

model solutions 8

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1.

solution. Recall that $[\alpha] + a[\beta] := [\alpha + a\beta]$. Hence, we must show that $\partial_k^*[\alpha + a\beta] = \partial_k^*[\alpha] + a\partial_k^*[\beta]$. Now given forms α' and β' for which

$$g^\# \alpha' = \alpha \quad g^\# \beta' = \beta.$$

Then we have that $g^\#(\alpha' + a\beta') = \alpha + a\beta$ be the linearity of $g^\#$. Then let α'' and β'' be such that

$$f^\# \alpha'' = d\alpha' \quad f^\# \beta'' = \beta'.$$

Then by construction

$$\partial^*[\alpha] = [\alpha''], \text{ and } \partial^*[\beta] = [\beta''].$$

But then $\partial^*[\alpha] + a\partial^*[beta] = [\alpha'' + a\beta'']$. And $\alpha'' + a\beta''$ satisfies

$$f^\#(\alpha'' + a\beta'') = d(\alpha' + a\beta')$$

by the linearity of $f^\#$ and d. Hence

$$\partial^*[\alpha + a\beta] = [\alpha'' + a\beta''],$$

and

$$\partial^*[\alpha + a\beta] = \partial^*[\alpha] + a\partial^*[\beta].$$

□

2.

solution. 1. The map $t \mapsto -1/t$ is smooth for $t \neq 0$, the map $\chi_{(0,\infty)}$ is smooth for $t \neq 0$ and $x \mapsto e^x$ is smooth for all x, so $\phi(t)$ is the product of two smooth functions whenever $t \neq 0$. Hence ϕ is smooth for $t \neq 0$. We intend to show that $\phi^{(n)}(t)$ is continuous and $\phi^{(n)}(0) = 0$. Claim: $\phi^{(n)}(t) = p_n(1/t)\chi_{(0,\infty)}e^{-1/t}$, for $t \neq 0$, where p_n is a polynomial.

This is easily seen by induction. It is true for $\phi^{(0)}(t) = \phi(t) = \chi_{(0,\infty)} p_0(1/t)e^{-1/t}$ where $p_0(x) = 1$. Then if true for n, then

$$\begin{aligned}\phi^{(n+1)}(t) &= -p_n'(1/t)\frac{1}{t^2}e^{-1/t}\chi_{(0,\infty)} + \frac{1}{t^2}p_n(1/t)e^{-1/t}\chi_{(0,\infty)} \\ &= \frac{1}{t^2}(p_n(1/t) - p_n'(1/t))e^{-1/t}\chi_{(0,\infty)}.\end{aligned}$$

Clearly $(p_n(1/t) - p_n'(1/t))/t^2$ is a polynomial in $1/t$.

Now we can show that $f(t) = \chi_{(0,\infty)} p(1/t)e^{-1/t}$ is continuous and equal to 0 at $t = 0$, and that

$$\frac{d}{dt}\,\chi_{(0,\infty)} p(1/t)e^{-1/t}\bigg|_{t=0} = 0.$$

It is well know that $e^{-x}p(x) \to 0$ as $x \to \infty$, so it is clear that $f(t)$ is continuous at $t = 0$. Now consider $\lim_{t\to 0}(f(t) - 0)/t$: on the left this is always 0, and on the right this is

$$\frac{1}{t}p(t)e^{-1/t} = \hat{p}(1/t)e^{-1/t}.$$

where $\hat{p}$ is the polynomial obtained by increasing the degree of each term of p by 1. Hence $f'(t) = 0$. Hence ϕ is smooth.

2. Define $\hat{\phi}(t) := \phi(t-a)\phi(b-t)$. Then $t < a$ $\hat{\phi}(t) = 0$, for $a < t < b$ $\hat{\phi}(t) > 0$ and for $t > b$ $\hat{\phi} = 0$. Then define

   $$\psi(t) := \frac{1}{\int_a^b \hat{\phi}\,dt}\int_{-\infty}^t \hat{\phi}(t)\,dt.$$

   The function ψ satisfies the desired properties and $\psi'(t) = \hat{\phi}(t)/\int_a^b \hat{\phi}\,dt$.

3. Let ψ be as above for $a = 1/2$ and $b = 1$. Then define

   $$\theta(y) := 1 - \psi(|y-x|/\varepsilon).$$

   The map $y \mapsto |y-x|$ is smooth for $y \neq x$, but by the chain rule

   $$D\theta(y) = -\psi'(|y-x|/\varepsilon)D|y-x|/\varepsilon.$$

   and this is zero for $|y-x|/\varepsilon < 1/2$, hence the map is smooth, non-negative, and support contained in the set $B(x,\varepsilon)$.

□

3.

solution. Define $K_\varepsilon \subset U$ to be the set $\{x \in U : |x| < 1/\varepsilon, d(x, \partial U) \geq \varepsilon\}$. For each $x \in K_1$ choose a radius $1/4 > r_x > 0$ such that $B(x, r_x) \subset U$ and $B(x, 2r_x) \subset V_i$ for some i. Then this forms a cover of K_ε. Choose a finite subcover, and denote this $B_k : k = 1, \dots, k_0$. Then Let B^0 denote the union $\bigcup_{k=1}^{k_0} B_k$. Then $B^0 \subset K_{1/2}$. Now suppose we have a finite cover $\mathcal{B}$ of $K_{2^{-p}}$ of balls with radius at most $2^{-(p+2)}$, satisfying properties (2) and (3). Then choose a finite subcover of $K^{2^{-(p+1)}} \setminus \bigcup \mathcal{B}$ of balls of radius at most $2^{-(p+3)}$ satisfying (2), called $\mathcal{B}'$ then $\mathcal{B} \cup \mathcal{B}'$ is a cover of $K_{2^{-p+1}}$ satisfying properties (2) and (3).

If we do this step again, we will get collection $\mathcal{B}''$ for which no member intersects any member of $\mathcal{B}$ when both members' radii are doubled.

In this way we can construct an increasing cover that covers $\bigcup_{\varepsilon > 0} K_\varepsilon = U$, and satisfies the desired properties. In fact this construction yields a stronger property: every point x has a neighbourhood U which intersects only finitely many balls, *i.e.* the cover is *locally finite.*

4.

solution. Using the preceding problem, we show have a refinement of $\mathcal{V}$ by balls B_k such that $2B_k \subset V_{i_k}$ for every k some i_k. For each ball B_k define θ_k to be the map with support $\overline{B}_k$, and hence support contained in $2B_k \subset V_{i_k}$. Then by construction the cover is locally finite, so take W to be a neighbourhood of x which only intersects finitely many balls. Then

$$\hat{\phi}(x) := \sum_k \theta_k(x),$$

is smooth because the sum is a locally finite sum of smooth functions. It is positive everywhere in U because $\bigcup_k B_k = U$, hence $\phi_k := \theta_k / \hat{\phi}$ is smooth and has support in $\overline{B}_k$, and $\sum_k \phi_k = 1$. □

5.

solution. 1. We must first show that the map is well defined:

$$\begin{aligned} f_*\sigma + f_*\partial\tau &= f_*\sigma + \partial f_* \\ f_*\sigma + f_*\partial\tau &\sim f_*\sigma. \end{aligned}$$

Now we must show linearity:

$$\begin{aligned} f_*([\sigma] + a[\tau]) &= f_*[\sigma + a\tau] \\ &= [f_*(\sigma + a\tau)] \\ &= [f_*\sigma + f_* a\tau] \\ &= [f_*\sigma + a f_*\tau] \\ &= [f_*\sigma] + a[f_*\tau] \\ &= f_*[\sigma] + a f_*[\tau]. \end{aligned}$$

2. Let σ be a one cycle, that is a map $\sigma \sum_i a_i\zeta_i$, where $\zeta_i[0,1] \to X$ for which σ. Then define $\widehat{H}^i := H^i(\zeta_i(\cdot),\cdot) : [0,1]^2 \to Y$. We define the 2-chain in $[0,1]^2$, given by $\square = \Delta_0 + \Delta_1$ where $\Delta_0 = [(0,0),(0,1),(1,0)]$ and $\Delta_1 = [(1,0),(0,1),(1,1)]$. Let $\tau_i = \widehat{H}^i_*\square$, then $\partial\tau_i = \widehat{H}^i_*\partial\square$:

$$\begin{aligned}\partial\square &= [(0,0),(0,1)] + [(0,1),(1,0)] + [(1,0),(0,0)] \\ &\quad + [(1,0),(0,1)] + [(0,1),(1,1)] + [(1,1)(1,0)] \\ &= [(0,0),(0,1)] + [(1,0),(0,0)] + [(0,1),(1,1)] + [(1,1)(1,0)]\end{aligned}$$

Now we can precompose with $\widehat{H}^i_*$ to yield

$$\widehat{H}_*\partial\square = H(\zeta_i(0),\cdot) - g_*\zeta_i + f_*\zeta_i + H(\zeta_i(1),1-\cdot).$$

And so

$$\sum_i a_i\partial\tau_i = \sum_i a_iH(\zeta_i(0),\cdot) - a_iH(\zeta_i(1),\cdot) + a_if_*\zeta_i - a_ig_*\zeta_i.$$

By σ being a cycle we have

$$\sum a_i(\zeta_i(0) - \zeta_i(1)) = 0.$$

from this it follows that

$$\sum_i a_iH(\zeta_i(0),\cdot) - a_iH(\zeta_i(1),\cdot) = 0.$$

$$\partial\sum_i a_i\tau_i = f_*\sigma - g_*\beta.$$

3. We note that clearly the zero chain $f_*\sigma(0) - g_*\sigma(0) = (\sigma(0),\cdot)$. Now for σ a 0- or 1-cycle $f_*\sigma \sim g_*\sigma$, so $f_*[\sigma] = g_*[\sigma]$, so $f_* = g_*$. The reason this is not shown for $k > 1$ is that the partition of the k-square $[0,1]^k$ into simplices is more complicated.

$\square$

6.

solution. We must first check that every chain (totally ordered subset) has an upper bound. Let $\{(W_\alpha, e_\alpha : I_\alpha \to W_\alpha) : \alpha \in A\}$ be such a chain. Then

$$\left(\bigcup_{\alpha\in A} W_\alpha, e_A : \bigcup_{\alpha\in A} I_\alpha \to \bigcup_{\alpha\in A} W_\alpha\right)$$

where $e_A|I_\alpha = e_\alpha$. Because the chain is totally ordered this function is well defined.

Hence we can apply Zorn's lemma and so there is a maximal element $(W, e : I \to W)$. Suppose that $V \setminus W \neq \varnothing$. Then choose $v \in V \setminus W$. Then $(W \oplus \mathbb{F}v, e' : \{I\} \cup I \to W \oplus \mathbb{F}v)$ where $e'(i) = e(i)$ for $i \in I$ and $e'(I) = v$. Clearly e' forms a basis as v is linearly independent from e. This element is clearly larger, contradicting maximality. Hence $(W, e : I \to W)$ is a basis. $\square$