Introduction to differential forms

model solutions 6

Jan Cristina

1.

solution. Consider the open cover of S^n by sets

$$U_{k+} := \{(x_1, \dots, x_{n+1}) \in S^n : x_k > 0\}$$
$$U_{k-} := \{(x_1, \dots, x_{n+1}) \in S^n : x_k < 0.\}$$

Any point on S^n has at least one coordinate different from 0, and hence either greater or less than 0.

Define

$$\phi_{k\pm} : U_{k\pm} \to B^n \quad (x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_{n+1}).$$

This is a homeomorphism with inverse

$$(x_1, \dots, x_n) \mapsto (x_1, \dots, x_{k-1}, \pm\sqrt{1 - \sum_{i=1}^{n} x_i^2}, x_k, \dots x_n.$$

For $U_{i\pm} \cap U_{j\pm}$ we have the transition map

$$\phi_{i\pm} \circ \phi_{j\pm}^{-1} : \phi_{j\pm}(U_{i\pm} \cap U_{j\pm}) \to \phi_{i\pm}(U_i \cap U_j)$$

has one component function given by $\sqrt{1 - \sum_l x_l^2}$, and the rest are given by projections $(x_1, \dots, x_n) \mapsto x_l$. Since $t \mapsto \sqrt(t)$ is smooth on $(0, \infty)$, the transition maps are smooth. The function $u(x_1, \dots, x_{n+1}) = x_{n+1}$ is smooth on S^n since

$$u \circ \phi_{n+1\pm}^{-1}(y_1, \dots, y_n) = \sqrt{1 - \sum_l y_l^2}$$

is smooth and

$$u \circ \phi_{k\pm}^{-1}(y_1, \dots, y_n = y_n$$

is smooth. □

2.

solution. It suffices to show that Q is homeomorphic to a smooth manifold. Then the smooth structure cane be pulled back via the homeomorphism. In this case we show that Q is homeomorphic to S^2.

The appropriate homeomorphism is given by translating and scaling Q to $\partial[-1,1]^3$, via $x \mapsto 2x - (1,1,1)$. Then because $\partial[-1,1]^3$ is compact and hausdorff we just need to construct a continuous bijection. But this is given by associating every point x on S^2 to the unique point on the ray from 0 to x intersecting $\partial[-1,1]^3$ □

3.

solution. We know that $T\mathcal{M} = \sqcup_{p\in M} T_pM$ for an M-manifold m. Given (U,x) a chart on M and $p \in M$. The vectors $\frac{\partial}{\partial x^1}|_p, \dots, \frac{\partial}{\partial x^n}|_p$ form a basis for T_pM, where

$$\left.\frac{\partial}{\partial x^i}\right|_p f = D_i(f \circ x^{-1})(x(p)).$$

This allows us to define a bijection $U \times \mathbb{R}^n \to TU = \sqcup_{p\in U} T_pM$, via $(p,v) \mapsto \sum_i v_i \partial_{x^i}|_p$. If we define a topology on TM by taking a base from the images of open sets under this bijection, the maps $\tau_U : U \times \mathbb{R}^n \to TU$ will be homeomorphisms if we can show that $W \subset TU$ is open only if $\tau_U^{-1}(W)$ is open It suffices to check this for another base element $W = \tau_V(W')$. Let us consider charts (U,x) and (V,y)

$$\tau_U^{-1} \circ \tau_V U \cap V \times \mathbb{R}^n \to U \cap V \times \mathbb{R}^n$$

$$(p,v) \mapsto (p,v')$$

where (p,v) is first mapped to $(p, \sum_i v_i \partial_{y^i})$ which by change of variables is equal to $\Phi(p,v) = (p, \sum_{i,j} v_i \partial_{y^j}(\phi^i)\partial_{x^i})$, where ϕ is the transition map $x \circ y^{-1}$ so is finally mapped to $(p, D\phi v)$, which is smooth in p and v. But then $\tau_U^{-1}(W) = \Phi(W')$ but Φ is a smooth homeomorphism, so this topology turns $T\mathcal{M}$ into a manifold, with smooth transition maps. We need only check that it is Hausdorff, and has a countable base.

It is Hausdorff because (p,ξ) can be separated from (p',ξ') by TU and TU' where U and U' separate p and p'. (p,ξ) can be separated from (p,ξ') by choosing appropriate sets under a local trivialisation about p.

A countable base exists, because each TU has a countable base, and $\mathcal{M}$ has a countable cover by charts.

□

4.

solution. Let $f : N \to M$ be a smooth map. We define $f_* : TN \to TM$ to be given by $(f_*\xi)(\phi) = \xi(\phi \circ f)$ where ξ is a derivation. In this case if ξ is a derivation at p then $f_*(\xi)$ is a derivation at $f(p)$ (this can be checked by the Leibniz rule). $\pi_M \circ f_* = f \circ \pi_N$ thus ι_* will suffice. We must show that f_* is smooth map, so let $U \subset N$ and $V \subset M$ be local trivialisation such that $p \in U$ and $f(p) \in V$. Then $\tau_U : TU \to U' \times \mathbb{R}^n$ and $\tau_V : TV \to V' \times \mathbb{R}^m$. Then let us see how f_* maps ∂_{x^i}

$$
\begin{aligned}
f_*\tau_U^{-1}(x(p), e_i) &= f_*\partial_{x^i}|_p\phi \\
&= \partial_{x^i}(\phi \circ f) \\
&= \frac{\partial}{\partial x^i}(\phi \circ f \circ x^{-1}) \\
&= \frac{\partial}{\partial x^i}(\phi \circ y^{-1} \circ y \circ f \circ x^{-1}) \\
&= \sum_{j=1}^{m} \frac{\partial F^j}{\partial x^i}\frac{\partial \phi \circ y^{-1}}{\partial y^j}, \\
&= \sum_{j=1}^{m} \frac{\partial F^j}{\partial x^i}\partial_{y^j}\phi
\end{aligned}
$$

where $F = y \circ f \circ x^{-1}$. This shows that

$$
\tau_V \circ f_* \circ \tau_U^{-1}(x(p), v) = \sum_{i=1}^{n}\sum_{j=1}^{m} v^i \frac{\partial F^j}{\partial x^i} e_j.
$$

We have that F is smooth by the assumption that f is smooth, and hence that $\tau_V \circ f_* \circ \tau_U^{-1}$ is smooth. Thus f_* is smooth. □

5.

solution. 1. $\alpha : P_p(M)/\sim \to T_pM$, $[\gamma] \mapsto \dot{\gamma}$, where $[\gamma] = \{\sigma : (-\delta, \delta) \to M : \sigma(0) = p,\ (x \circ \sigma)'(0) = (x \circ \gamma)'(0) \text{ for all charts } x\}$. Suppose $\sigma, \gamma \in P_p(M)$ are similar, $\gamma \sim \sigma$. Then we have $(x \circ \sigma)'(0) = (x \circ \gamma)'(0)$ for all coordinates x.

$$
\begin{aligned}
\dot{\gamma} &= \sum_{i=1}^{n} \dot{\gamma}(x_i)\partial_{x^i}|_p \\
&= \sum_{i=1}^{n} (x^i \circ \gamma)'(0)\partial_{x^i}|_p \\
&= \sum_{i=1}^{n} (x^i \circ \sigma)'(0)\partial_{x^i}|_p \\
&= \dot{\sigma}.
\end{aligned}
$$

Hence $\gamma \sim \sigma \Rightarrow \dot{\gamma} = \dot{\sigma}$. Now given $\xi \in TM$ take $(0, v) = \tau_U \xi)$, and $p = \pi(\xi)$, then let $\gamma(t) = x^{-1}(0 + tv)$. Then $\dot{\gamma} = \xi$ clearly.

Suppose $\dot{\gamma} = \dot{\sigma}$ then

$$\begin{aligned}(x \circ \gamma)'(0) &= ((x_1 \circ \gamma)'(0), \ldots, (x_n \circ \gamma)'(0)) \\ &= ((x_1 \circ \sigma)'(0), \ldots, (x_n \circ \sigma)'(0)) \\ &= (x \circ \sigma)'(0).\end{aligned}$$

Hence the map is injective and surjective, and so an isomorphism.

2. This is a straight-forward application of problem 1-3. We just note that the derivations of $\mathbb{R}^n$ are given by partial differentiation, whose basis at a point is given by ∂_{x^i}.

□

6.

solution. The end result we are striving for is

$$\Theta^{-1} T S^{n-1} = \{(p, v) : p \in S^{n-1}, \langle p, v \rangle = 0\}.$$

Let $p = (p_1, \ldots, p_n)$ and assume without loss of generality that and $p_n > 0$ (otherwise take $k = n$ and reorder). Let $\phi : U_{n+} \to B^{n-1}$ be the map $(p_1, \ldots, p_n) \mapsto (p_1, \ldots, p_{n-1})$. Then $\psi : B^{n-1} \to S^{n-1}$ $p \mapsto (p, \sqrt{1 - |p|^2})$.

Set $v = \sum_{k=1}^{n-1} v_k \partial_{\phi^k}|_p$. Let $\gamma : (-\delta, \delta) \to \mathbb{R}^n$ be given by $\gamma(t) = p + D\phi^{-1} \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}$ and $\alpha_v : (-\delta, \delta) \to B^{n-1}$ $t \mapsto \phi(p) + t \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}$. Then

$$\begin{aligned}i_* v f &= \sum_{k=1}^{n-1} i_* (\partial_{\phi^k})_p f \\ &= \sum_{k=1}^{n-1} v_k \partial_{y^k}|_p (f \circ i \circ \phi^{-1})_{\phi(p)} \\ &= (f \circ \phi^{-1} \circ \alpha_v)'(0),\end{aligned}$$

by the chain rule for $f \in C^\infty(\mathbb{R}^n)$ and the fact that $\alpha_v'(0) = (v_1, \ldots, v_n)^t$. Thus

$$\begin{aligned}i_* v f &= (f \circ \phi^{-1} \circ \alpha_v)'(0) && (1) \\ &= \nabla f (D\Phi^{-1}) \alpha_v'(0) && (2) \\ &= \nabla f \gamma'(0) && (3) \\ &= (f \circ \gamma)'(0) = \dot{\gamma}(f). && (4)\end{aligned}$$

Where $\gamma = \gamma_{\tilde{v}}$ and $\tilde{v} = D\phi^{-1}\begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \end{pmatrix}$. Since

$$D\Phi^{-1}(y) = \begin{pmatrix} I^{n-1)\times(n-1)} \\ u \end{pmatrix}$$

where $u = -(y^1, \ldots, y_{n-1})/\sqrt{1-|y|^2}$, we have that

$$\tilde{v} = (v_1, \ldots, v_{n-1}, -\sum_{k=1}^{n-1} v_k p_k / p_n),$$

where $p = \phi^{-1}(y)$. Then

$$\tilde{v} \cdot p = 0.$$

□