Introduction to differential forms

model solutions 3

Jan Cristina

1.

solution. □

Let $\{v_1, \dots, v_q\}$ and $\{w_1, \dots, w\_p\}$ be collections of vectors in an n-dimensional vector space V. Let v_I denote $v_{i_1} \wedge \cdots \wedge v_{i_k}$ for $I = \{i_1, \dots, i_k\} \subset \{1, \dots, q\}$, and similarly for w_I. Now Let us assume that

$$\sum_I a_I v_I = \sum_J b_J w_J.$$

Then we wish to show that

$$f_* \sum_I a_I v_I = f_* \sum_J b_J w_J.$$

Without loss of generality we may assume that the v_i are a basis and $q = n$ (while the w_i need not be). Because of this, we may express $w_i = \sum_{j=1}^n T_{ij} v_j$, $i = 1, \dots, p$. Then we know that

$$\sum_I a_I v_I = \sum_{J,L} b_J T_{J,L} v_L$$

where $T_{J,L} = \sum_{\sigma \in S_k} \text{sgn}\,(I, [n]) \text{sgn}\,(L, [q]) \text{sgn}\,(\sigma) t_{j_1 l_{\sigma(1)}} \cdots t_{j_k l_{\sigma(k)}}$, and $\text{sgn}\,(I, [n])$ denotes the sign of the permutation that takes the j to i_j for $1 \le j \le k$, and puts the remaining numbers in increasing order. But then $a_I = \sum_J b_J T_{J,L}$. Now we note

$$\sum_I a_I f_*(v_I) = \sum_{I,J} b_J T_{J,I} f_*(v_I).$$

But we can pull the coefficients $T_{J,I}$ into the push forward, to get that this is

$$\begin{aligned} &= \sum_J b_J f \left(\sum_{l_1=1}^n T_{j_1 l_1} v_{l_1} \right) \wedge \cdots \wedge f \left(\sum_{l_k=1}^n T_{j_k l_k} \right) \\ &= \sum_J b_J f_*(w_J). \end{aligned}$$

As for the second property, we note that by definition $\underline{\omega}(w_1 \wedge \cdots w_k) = \omega(w_1, \dots, w_k)$, and we note that

$$\begin{aligned} f^*\omega(v_1, \dots, v_k) &= \omega(f(v_1), \dots, f(v_k)) \\ &= \underline{\omega}(f(v_1) \wedge \cdots \wedge f(v_k)) \\ &= \underline{\omega}(f_*(v_1 \wedge \cdots \wedge v_k)). \end{aligned}$$

2.

solution. Let $w_1 = v_1, \dots, w_m = v_m$ denote a basis of W and $v_{m+1}, \dots, v_n$ complete these to a basis of V. First to show that the map ι_* is injective: suppose $\sum_I \alpha_I \iota_*(w_I) = 0$, then $\sum_I \alpha_I v_I = 0$ but the v_I are a basis of $\bigwedge_k V$, so $\alpha_I = 0$, and $\sum_I \alpha_I w_I = 0$.

Now for surjectivity of ι^*. Let $\omega_1, \dots, \omega_m$ denote a dual basis of $w_1, \dots, w_m$, and let $\nu_1, \dots, \nu_n$ denote a basis dual to the v_i. Then consider $\iota^*(v_I)$ for $I \subset [m] \subset [n]$.

$$\begin{aligned} \iota^*(\nu_I)(w_{i_1}, \dots, w_{i_k}) &= \nu_I(\iota(w_{i_1}), \dots, \iota(w_{i_k})) \\ &= \nu_I(v_{i_1}, \dots, v_{i_k}) \\ &= \begin{cases} 1 & \text{if } I = \{i_1, \dots, i_k\} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

But in this case $\iota^*\nu_I = \omega_I$, and I was arbitrary, so ι^* is surjective. □

3.

solution.

1. Let ω_i be a (not necessarily standard) basis for $\mathbb{R}^{4*}$. Then by the condition, we have that $f^*(\omega_1) \wedge f^*(\omega_2) = \omega_1 \wedge \omega_2)$. If we write $f^*\omega_i = f_i + g_i$ for $f_i \in \text{span}\{\omega_1, \omega_2\}$, and $g_i \in \text{span}\{\omega_3, \omega_4\}$, then

   $$f^*(\omega_1) \wedge f^*(\omega_2) = f_1 \wedge f_2 + f_1 \wedge g_2 + g_1 \wedge f_2 + g_1 \wedge g_2.$$

   All of the terms are linearly independent, and excluding possibly $f_1 \wedge f_2$ are linearly independent from $\omega_1 \wedge \omega_2$, hence they are 0. But if two one forms α and β wedge to 0, then they are scalar multiples of one another. But the f_i and g_i are in linearly independent spaces, and so the g_i are 0.

   But then $f^*\text{span}\{\omega_i, \omega_j\} \subset \text{span}\{\omega_i, \omega_j\} =: P_{ij}$ for all $i \neq j$.

2. If we intersect P_{12} and P_{23} we get $\mathbb{R}\omega_2$, but because $f^*P_{12} = P_{12}$ and $f^*P_{23} = P_{23}$ we get that $f^*\omega_2 = \lambda\omega_2$. But our choice of basis was arbitrary, so we have that $f^*\omega = \lambda\omega$ for all ω. Now suppose η and ω are linearly independent then if $f^*\omega = \lambda\omega$ and $f^*\eta = \mu\eta$, and $f^*(\omega + \eta) = \nu(\omega + \eta)$. But $f^*(\omega + \eta) = \lambda\omega + \nu\eta$, hence $\mu = \nu = \lambda$, and $f^* = \lambda\text{id}$. If this is the case then we have that $f^*\eta \wedge f^*\omega = \lambda^2\eta \wedge \omega = \eta \wedge \omega$ from which we get that $\lambda = \pm 1$.

3. $f = \pm\text{id}$.

4. Here is a more algebraic proof that uses the idea of an annihilator. Let $e_1, \ldots, e_4$ be an arbitrary basis with dual $\varepsilon_1, \ldots, \varepsilon_4$. Let α be a 2-form, and define the annihilator of α,

$$A(\alpha) := \{v \in \mathbb{R}^4 : \forall w \in \mathbb{R}^4\, \alpha(v, w) = 0\}.$$

That is $A(\alpha)$ is the set of vectors which always go to 0. It is trivial to see that $A(\alpha)$ is a linear subspace for all α.

Now we make the following claim: for i, j, k, l any permutation of the numbers $1, 2, 3, 4$, that

$$A(\varepsilon_i \wedge \varepsilon_j) = \text{span}\,\{e_k, e_l.$$

We show the case $i = 1, j = 2mk = 3, l = 4$. Consider

$$\begin{aligned}\varepsilon_1 \wedge \varepsilon_2(\alpha e_1 + \beta e_2, -\beta e_1 + \alpha e_2) &= \alpha^2 \varepsilon_1(e_1)\varepsilon_2(e_2) - \beta(-\beta)\varepsilon_2(e_2)\varepsilon_1(e_1)\\ &= \alpha^2 + \beta^2,\end{aligned}$$

which is nonzero for $\alpha e_1 + \beta e_2$ not equal to 0. Then trivially e_3 and e_4 are in $A(\varepsilon_1 \wedge \varepsilon_2)$, but then they span a two dimensional subspace $\text{span}\,\{e_3, e_4\} \subset A(\varepsilon_1 \wedge \varepsilon_2)$ which is complementary to $\text{span}\,\{e_1, e_2\}$, which intersects $A(\varepsilon_1 \wedge \varepsilon_2)$ only at 0, so $A(\varepsilon_1 \wedge \varepsilon_2) = \text{span}\,\{e_3, e_4\}$.

Now let because $f^*(\alpha) = \alpha$, for any two form, we know that $f(A(\alpha)) \subset A(\alpha)$ for any two form. Now consider that $e_1 \in A(\varepsilon_i \wedge \varepsilon_j)$ for $i \neq 1 \neq j$. But then $f(e_1) \in A(\varepsilon_i \wedge \varepsilon_j)$. But the only vectors that are in

$$A(\varepsilon_2 \wedge \varepsilon_3) \cap A(\varepsilon_3 \wedge \varepsilon_4) \cap A(\varepsilon_4 \wedge \varepsilon_2),$$

are those of the form λe_1. Hence $f(e_1) = \lambda e_1$. But our choice of basis was arbitrary, so $f(v) = \lambda_v v$ for every v, so every vector is an eigenvector. Then proceed as before. □

4.

solution. 1. Let $f : V^k \to W$ be alternating and multilinear. Define $\underline{f} : \bigwedge_k V \to W$ by $\underline{f}(v_1 \wedge \cdots \wedge v_k) := f(v_1, \ldots, v_k)$. Before we can say that this defines a unique map, it is worth mentioning that it actually defines a map. To do this we will show that for a given basis, $e_1, \ldots, e_n$ the map defined in that way extends to all k-vectors. This will show uniqueness of the map, as the k-vectors e_I define a basis for $\bigwedge_k V$. Let $v_i = \sum_j a_{ij} e_j$. Then

$$v_{[k]} = \sum_J a_{[k],J} e_J,$$

where $a_{[k],J}$ are defined as in the solution to problem 1. Then by a linear extension, we have that

$$\begin{aligned}\underline{f}(v_{[k]}) &= \sum_J a_{[k],J}\underline{f}(e_J)\\ &= \sum_J a_{[k],J} f(e_{j_1},\ldots,e_{j_k}).\end{aligned}$$

Now we can apply multilinearity of f to arrive at

$$\begin{aligned}\underline{f}(v_{[k]}) &= f(\sum_{j_1} a_{1j_1}e_{j_1},\ldots,\sum_{j_1} a_{kj_k}e_{j_k})\\ &= f(v_1,\ldots,v_k).\end{aligned}$$

Hence the map is well defined and uniquely defined by extending linearly from a basis.

2. First we construct the multilinear map $\vartheta : V^k \to \bigwedge_k V$ $\vartheta : (v_1,\ldots,v_k) \to v_1 \wedge \cdots \wedge v_k$. Then by the unique lifting property of X and $\bigwedge_k V$, we have maps $\underline{\theta} : \bigwedge_k V \to X$, and $\underline{\vartheta} : X \to \bigwedge_k V$. But with these we have that $\underline{\theta} \circ \vartheta = \theta$, But $\underline{\vartheta} \circ \theta = \vartheta$, so

$$\underline{\vartheta} \circ \underline{\theta} \circ \vartheta = \vartheta.$$

But ϑ is surjective onto a basis of $\bigwedge_k V$, so $\underline{\vartheta} \circ \underline{\theta} = \mathrm{id}$.

Similarly the map $\underline{\theta} \circ \underline{\vartheta} \circ \theta = \theta$. But $\mathrm{id} : X \to X$ is the unique lift of the multilinear map θ, so $\underline{\theta} \circ \underline{\vartheta} = \mathrm{id}$. Hence $X \cong \bigwedge_k V$.

□

5.

solution. Suppose that f is invertible otherwise both sides of the equation are 0. Here it is sufficient to examine simple functions. Let $u = \sum_i a_i \chi_{E_i}$. Then

$$\begin{aligned}\int_{\mathbb{R}^n} u \circ f |J_f|\, d\mathcal{L}^n &= \int_{\mathbb{R}^n} \sum_{a_i} \chi_{E_i} \circ f |\det f|\, d\mathcal{L}^n\\ &= \int_{\mathbb{R}^n} \sum_{a_i} \chi_{f^{-1}(E_i)} |\det f|\, d\mathcal{L}^n\\ &= \sum_i a_i |f^{-1}(E_i)||\det f|\\ &= \sum_i a_i |E_i||\det f^{-1}||\det f|\\ &= \sum_i a_i |E_i|\\ &= \int_{f(\mathbb{R}^n)} u\, d\mathcal{L}^n.\end{aligned}$$

□

6.

solution. A covering map $\pi : B \to X$ is an open map between connected spaces for which for every $x \in X$, there is a $U \subset X$ containing x, such that $\pi^{-1}(U) = \sqcup_i U_i$, and $\pi|U_i$ is a homeomorphism.

1. The set $\sigma([0,1])$ is compact, so it may be covered by a finite collection of covering sets U^i. Then every point $t \in [0,1]$ has a neighbourhood $[t-\varepsilon_t, t+\varepsilon_t]$ contained entirely in $\sigma^{-1}(U^i)$. Cover I with the sets $(t-\varepsilon_t, t+\varepsilon_t)$, then there is a finite subcover (t_i, s_i). Arranging eng points in order from least to largest and relabeling τ_i, we get intervals $[\tau_i, \tau_{i+1}]$ whose image under σ is contained entirely in U^i for some i.

   Now define inductively $\gamma_i : [\tau_i, \tau_{i+1}] \to X$, to be
   $$\gamma_i(t) = (f|U^k)^{-1} \circ \sigma(t),$$
   where U^k contains $\sigma([\tau_i, \tau_{i+1})$ and U^k_l contains $\gamma_{i-1}(\tau_i)$. Then set $\gamma_0(0) = x_0$, and we are done. The conjoined curve $\tilde{\sigma} = \gamma_N * \cdots * \gamma_0$ satisfies the desired properties at each step of the induction, hence satisfies it in general.

2. Comence by lifting each path the path $H(s,0)$ to $\tilde{H}(s,0)$ by the unique lift that $H(s,0) = x_0$. Then for each s define $\tilde{H}(s,t)$ to be the lift of the curve $t \mapsto H(s,t)$ starting at x_0. Then by construction $f \circ \tilde{H} = H$.

   Now we must show that $\tilde{H}$ is continuous. To do this let U^i denote a finite cover of $H(I^2)$ by covering sets. Now set for each pair $(t,s) \in I^2$ denote $[t-\varepsilon, t+\varepsilon] \times [s-\varepsilon, s+\varepsilon]$ a set contained entirely in som $H^{-1}(U^i)$ (ε depends on s and$t$). Then we can take a finite subcover with these types of sets, $[t_i^1, t_i^2] \times [s_i^1, s_i^2]$. Now by placing all of the s_i^j and t_i^j in order on their respectively intervals and relabeling in that order, we get rectangles $[t_i, t_{i+1}] \times [s_j, s_{j+1}]$ such that $\mathcal{H}([t_i, t_{i+1}] \times [s_j, s_{j+1}]) \subset U^k$ for some k.

   Now we claim that $\tilde{H}(s,t) = (f|U^k_l)^{-1}(\mathcal{H})$ for some U^k_l and for all t and s. To see this we do induction on s and t. Suppose the claim is true for all $t \le t_i$ and $s \le s_j$. Then choose $t \in [t_i, t_{i+1}]$, and $s \in [s_{j'}, s_{j'+1}]$ for $j'+1 \le j$. Then $\mathcal{H}([t_i, t_{i+1}] \times [s_{j'}, s_{j'+1}]) \subset U^k$ for some k. But then $\tilde{H}(s, t_i) = (f|U^k_l)^{-1}(\sigma(s, t_i))$ by the inductive hypothesis, and $\tilde{H}(s, \cdot)$ is a lift of $H(s, \cdot)$, so $\tilde{H}(s,t) = (f|U^k_l)^{-1}(\sigma(s,t))$ for all $t < t_i$. Once t has been increased, s can be increased in a similar vein. Hence $\tilde{H}$ is locally continuou, but then it is globally continuous.

3. To show this we must first show that if γ and γ' are two curves, and $f \circ \gamma \sim f \circ \gamma'$ then $\gamma \sim \gamma'$. But this is easy with the previous, because let H be a homotopy between $f \circ \gamma$ and $f \circ \gamma'$ starting at $f(x_0)$, then $\tilde{H}$ at x_0 is a homotopy between γ and γ'.

□