Introduction to differential forms

model solutions 10

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1.

solution. 1. The maps $I_\#$ and $J_\#$ are chain maps if they commute with the exterior derivative. But $I_k = (i_1^*, i_2^*)$, and hence

$$\begin{aligned} I_k \circ d &= (i_1^* d, i_2^* d) \\ &= (d i_1^*, d i_2^*) \\ &= d \circ I_{k-1}. \end{aligned}$$

Similarly

$$\begin{aligned} J_k \circ d &= (j_1^* - j_2^*) d \\ &= d(j_1^* - j_2^*) \\ &= d J_{k-1}. \end{aligned}$$

2. The goal is to show that ω_1 and ω_2 are in fact smooth. The proof for ω_2 is essentially the same as that for ω_1, so we will only show it for ω_1. For $x \in U_1 \cap U_2$ and $U_1 \setminus \overline{U}_2$ this is immediate, as ω_1 is the product of two smooth functions. Now choose $x \in \partial U_2 \cap U_1$. Then $\operatorname{dist}(x, \operatorname{spt}\phi_2) > 0$, hence $\phi_2|_{B(x,\varepsilon)} \equiv 0$ for some ε. Consequently $\omega\phi_2 \equiv 0$ in $B(x,\varepsilon)$, hence ω_1 is smooth about $x \in \partial U_2 \cap U_1$.

□

2.

solution. 1. Let $U = \mathbb{R}^3 \setminus \{(x,y,0) \in \mathbb{R}^3 : y \leq 0\}$ and $V = \mathbb{R}^3 \setminus \{(x,y,0) \in \mathbb{R}^3 : y \geq 0\}$. Then $\mathbb{R}^3 \setminus L = U \cup V$, U and V are star shaped, and $U \cap V = \mathbb{R}^3 \setminus \{(x,y,0) \in \mathbb{R}^3\}$, is a disjoint union of two star shaped domains. By Mayer-Vietoris

$$\cdots \to H^{k-1}(U \cap V) \to H^k(U \cup V) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \to \cdots$$

From this we get for $k = 0, 1$,

$$0 \to H^0(U \cup V) \to \mathbb{R} \oplus \mathbb{R} \xrightarrow{J_0^*} \mathbb{R}^2 \to H^1(U \cup V) \to H^1(0).$$

We know that $H^0(U_1)$ is given by the locally constant functions, so the kernel of J_0^* is given by $\{(c,c) : c \in \mathbb{R}\}$ and so $H^0(U \cup V) = \mathbb{R}$. From this it also follows that $H^1(U \cup V) = \mathbb{R}$. For $k > 1$

$$0 \to H^k(U \cup V) \to 0 \oplus 0 \to 0,$$

so $H^k(U \cup V) = 0$.

2. The set $\mathbb{R}^3 \setminus L$ is homotopic to $\mathbb{R}^2 \setminus \{0\}$, whose cohomology was calculated in lectures as $H^0(\mathbb{R}^2 \setminus \{0\}) = \mathbb{R} = H^1(\mathbb{R}^2 \setminus \{0\})$.

3. We can use theorem "A", setting $A = \{\mathbb{R} \times \{0\}$, then for $k \geq 1$

$$H^{k+1}(\mathbb{R}^3 \setminus A \times \{0\}) \cong H^k(\mathbb{R}^2 \setminus A)$$

and

$$H^0(\mathbb{R}^3 \setminus A \times \{0\}) \cong \mathbb{R} \quad H^1(\mathbb{R}^3 \setminus A \times \{0\}) H^0(\mathbb{R}^2 \setminus A)/\mathbb{R}.$$

Going down one dimension yields $\mathbb{R}^2 \setminus A$ is a disjoint union of two star shaped domains so $H^k(\mathbb{R}^2 \setminus A) = 0$ for $k \geq 1$ and $H^0(\mathbb{R}^2 \setminus A) \cong \mathbb{R}^2$.

□

3.

solution. We once again use excision noting that $\mathbb{R}^2 \setminus S = D \cup A$ where D is the unit disk and A is homotopic to $\mathbb{R}^2 \setminus \{0\}$, hence

$$H^*(\mathbb{R}^2 \setminus S) = (\mathbb{R}^2, \mathbb{R}, 0, 0, \cdots).$$

Hence

$$H^*(\mathbb{R}^3 \setminus S) = (\mathbb{R}, \mathbb{R}, \mathbb{R}, 0, \cdots),$$

$$H^*(\mathbb{R}^n \setminus S) = (\mathbb{R}, \overbrace{0, \cdots, 0}^{n-3 \text{ times}}, \mathbb{R}, \mathbb{R}, 0, 0, \cdots).$$

□

4.

solution. We note first that T is a connected open set (as is $T \setminus S$), so their zeroth cohomology groups are both $\mathbb{R}$. We use Mayer-Vietoris for $U = T$, and $V = \mathbb{R}^3 \setminus S$, Then $T \setminus S = U \cap V$, and $U \cup V = \mathbb{R}^3$:

$$\begin{aligned} &0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R} \to 0 \to H^1(T) \oplus \mathbb{R} \\ &\quad \to H^1(T \setminus S) \to 0 \to H^2(T) \oplus \mathbb{R} \to H^2(T \setminus S) \to 0 \to H^3(T) \to H^3(T \setminus S) \to 0. \end{aligned}$$

Let $S^+ = \{\theta \in S : \Re(\theta > -1/2)$ and $S^- = \{\theta \in S : \Re(\theta) < 1/2\}$. Let $U = \phi(B^2 \times S^+)$ and $V = \phi(B^2 \times S^-)$. Then U is homeomorphic to V, and are both diffeomorphic to balls in $\mathbb{R}^3$, and $U \cap V$ is diffeomorphic to the disjoint union of two balls in $\mathbb{R}^3$, so

$$0 \to \mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}^2 \to H^1(T) \to 0 \to 0 \to H^2(T) \to 0 \to 0 \to H^3(T) \to 0.$$

Hence $H^1(T) \cong \mathbb{R}$ and $H^k(T) = 0$ for $k \geq 2$. Hence

$$H^2(T \setminus S) = (\mathbb{R}, \mathbb{R}^2, \mathbb{R}, 0, 0, \cdots).$$

□

5. l

solution. Let $i^* : \Omega_c^k(U_1 \cap U_2) \to \Omega_c^k(U_l)$, and $j_l^* : \Omega_c^k(U_l) \to \Omega_c^k(U_1 \cup U_2)$. Then define $I_k := i_1^* \oplus i_2^*$, and $J_k := j_1^* - j_2^*$. First we must show that I_k is injective: assume $I_k(\omega) = 0$, then $i_1^*(\omega) = 0 = i_2^*(\omega)$. Then $\omega = 0$. Hence I_k is injective.

Now take $\omega \in \Omega_c^k(U_1 \cup U_2)$. Let ϕ_1, ϕ_2 be a partition of unity subordinate to U_1 and U_2. Then $\phi_1\omega, -\phi_2\omega)$ is in $\Omega_c^k(U_1) \oplus \Omega_c^k(U_2)$, and maps under J_k to ω.

Lastly suppose $J_k(\omega_1, \omega_2) = 0$, then $j_1^*(\omega) = j_2^*(\omega_2)$. But then ω_1 and ω_2 are compactly supported in $U_1 \cap U_2$, and hence equal. So $(\omega_1, \omega_2) = I_k(\omega)$. □

6.

solution. Let $U' = U \setminus A$, and for each $x \in U$ choose an open set U_x such that $|f(z) - f(x)| < \varepsilon'(z)$ for all $z \in U_x$. If $x \in U'$ choose $U_x \subset U'$, and if $x \in A$ choose $U_x \subset W$. Then let choose a locally finite partition of unity ϕ_i supported in $U_i \in U_{x_i}$. Let $\phi_W = \sum_{x_i \in A} \phi_i$. Then $\phi_W | A \equiv 1$, and $\mathrm{spt}(\phi_W) \subset W$. This is because if $x \in \mathrm{spt}(\phi_i)$ implies $U_i \cap A \neq \varnothing$, which implies $x_i \in A$. But then $\sum_{\mathrm{spt}(phi_i) \ni x} \phi_i(x) = 1$ for every x, and so $\phi_W(x) = 0$. Furthermore if $x_i \in A$, then $U_i \subset W$ so $\mathrm{spt}(\phi_i) \subset W$.

Now define

$$g(x) = \phi_W(x) f(x) + \sum_{x_i \in U'} \phi_i(x) f(x_i).$$

Then g is smooth because $f(x)$ is smooth on the support of ϕ_W, and ϕ_i is smooth.

Now

$$\begin{aligned}
|g(x) - f(x)| &\leq \phi_W f(x) + \sum_{x_i \in U'} \phi_i(x) f(x_i) - \phi_W f(x) - \sum_{x_i \in U'} \phi_i(x) f(x) \\
&\leq \sum_{x_i \in U'} \phi_i(x) |f(x_i) - f(x)| \\
&\leq \sum_{x_i \in U'} \phi_i(x) \varepsilon'(x) \\
&\leq \varepsilon'(x).
\end{aligned}$$

Now choose $\varepsilon'(x) := \min\{\varepsilon(x), \mathrm{dist}(f(x), V^c)/2)$, then

$$\begin{aligned} \mathrm{dist}(g(x), V^c) &\geq \mathrm{dist}(f(x), V^c) - |g(x) - f(x)| \\ &\leq \mathrm{dist}(f(x), V^c)/2 > 0. \end{aligned}$$

Hence $g(x) \in V$, and $g : U \to V$.

$\square$