**Introduction to Differential forms**
**Spring 2011**
**Exercise 11 (for Wednesday Apr. 20)**

$\star$**1.** Let $U_1 = \mathbb{R}^2 \setminus [0,\infty) \times \{0\}$ and $U_2 = \mathbb{R}^2 \setminus (-\infty, 0] \times \{0\}$. Find generators for cohomologies $H^k(\mathbb{R}^2 \setminus \{0\})$ and find formulas for $\partial_k^*\colon H^k(U_1 \cap U_2) \to H^{k+1}(U_1 \cup U_2)$ in terms of these generators for all $k \geq 0$.

**2.** Let $f\colon \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ be the map (using complex notation) $z \mapsto z^2$. Calculate $f^*\colon H^1(\mathbb{R}^2 \setminus \{0\}) \to H^1(\mathbb{R}^2 \setminus \{0\})$.

$\star$**3.** Let

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A_* & \xrightarrow{f^\#} & B_* & \xrightarrow{g^\#} & C_* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha^\#} & & \downarrow{\scriptstyle \beta^\#} & & \downarrow{\scriptstyle \gamma^\#} & & \\
0 & \longrightarrow & \tilde{A}_* & \xrightarrow{\tilde{f}^\#} & \tilde{B}_* & \xrightarrow{\tilde{g}^\#} & \tilde{C}_* & \longrightarrow & 0
\end{array}$$

be a commutative diagram of chain complexes and chain maps with exact rows.

(1) Show that the diagram

$$\begin{array}{ccc}
H^k(C_*) & \xrightarrow{\partial_k^*} & H^{k+1}(A_*) \\
\downarrow{\scriptstyle \gamma^*} & & \downarrow{\scriptstyle \alpha^*} \\
H^k(\tilde{C}_*) & \xrightarrow{\tilde{\partial}_k^*} & H^{k+1}(\tilde{A}_*)
\end{array}$$

commutes.

(2) Conclude that the diagram

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H^k(A_*) & \xrightarrow{f^*} & H^k(B_*) & \xrightarrow{g^*} & H^k(C_*) & \xrightarrow{\partial_k^*} & H^{k+1}(A_*) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \alpha^*} & & \downarrow{\scriptstyle \beta^*} & & \downarrow{\scriptstyle \gamma^*} & & \downarrow{\scriptstyle \alpha^*} & & \\
\cdots & \longrightarrow & H^k(\tilde{A}_*) & \xrightarrow{\tilde{f}^*} & H^k(\tilde{B}_*) & \xrightarrow{\tilde{g}^*} & H^k(\tilde{C}_*) & \xrightarrow{\tilde{\partial}_k^*} & H^{k+1}(\tilde{A}_*) & \longrightarrow & \cdots
\end{array}$$

commutes.

$\star$**4.** Let $R\colon \mathbb{R}^{n+1} \setminus A \times \{0\} \to \mathbb{R}^{n+1} \setminus A \times \{0\}$ be the map $(x_1, \ldots, x_{n+1}) \mapsto (x_1, \ldots, x_n, -x_{n+1})$. Let sets $U_1$ and $U_2$ and maps $R_0, R_1, R_2$ be as in the proof of Theorem A in the lecture notes.

(1) Show that the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^k(\mathbb{R}^{n+1} \setminus A \times \{0\}) & \longrightarrow & \Omega^k(U_1) \oplus \Omega^k(U_2) & \longrightarrow & \Omega^k(U_1 \cap U_2) & \longrightarrow & 0 \\
 & & \downarrow R^* & & \downarrow \rho & & \downarrow -R_0^* & & \\
0 & \longrightarrow & \Omega^k(\mathbb{R}^{n+1} \setminus A \times \{0\}) & \longrightarrow & \Omega^k(U_1) \oplus \Omega^k(U_2) & \longrightarrow & \Omega^k(U_1 \cap U_2) & \longrightarrow & 0,
\end{array}$$

where $\rho(\omega_1, \omega_2) = (R_1^*\omega_2, R_2^*\omega_1)$, commutes.

(2) Show that $\tilde{\partial}^* \circ R_0^* = -R^* \circ \partial^*$ for every $k \geq 1$.

(3) Conclude that $R^* = -\mathrm{id}\colon H^k(\mathbb{R}^{n+1} \setminus A \times \{0\}) \to H^k(\mathbb{R}^{n+1} \setminus A \times \{0\})$ for $k \geq 1$.

**5.** Let $f\colon \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ be the map $x \mapsto -x$. Show that $f^* = (-1)^n \mathrm{id}\colon H^{n-1}(\mathbb{R}^n \setminus \{0\}) \to H^{n-1}(\mathbb{R}^n \setminus \{0\})$.

**6.** Show that the map $g$ in the proof of Brouwer's fixed point theorem has the formula $g(x) = x + t(x)u(x)$, where $u(x) = (x - f(x))/|x - f(x)|$ and $t(x) = -\langle x, u(x)\rangle + (1 - |x|^2 + \langle x, u(x)\rangle^2)^{1/2}$.