Introduction to Differential forms
Spring 2011
Exercise 9 (for Wednesday Apr. 6)

$\star$**1.** Suppose that $0 \longrightarrow A^* \xrightarrow{f^\#} B^* \xrightarrow{g^\#} C^* \longrightarrow 0$ is an exact sequence of chain complexes and chain maps. Show that, for every $k \in \mathbb{Z}$, the map $\partial_k^* \colon H^k(C^*) \to H^{k+1}(A^*)$, constructed in the lecture notes, is linear.

2.

(i) Show that the function $\varphi \colon \mathbb{R} \to \mathbb{R}$, $\varphi(t) = \chi_{(0,\infty)} e^{-1/t}$, is C^∞-smooth.

(ii) Show that,for every $-\infty < a < b < \infty$ there exists a C^∞-smooth function $\psi \colon \mathbb{R} \to [0,1]$ so that $\psi(t) = 0$ for $t < a$ and $\psi(t) = 1$ for $t > b$.

(iii) Show that for every $x \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists a C^∞-smooth function $\theta \colon \mathbb{R}^n \to [0,\infty)$ so that $B^n(x,\varepsilon) = \theta^{-1}(0,\infty)$.

3. Let $U \subset \mathbb{R}^n$ be an open set and $\mathcal{V} = \{V_i\}_{i\in I}$ an open cover of U. Show that there exists a sequence $B_k = B^n(x_k, r_k)$ of open balls so that (1) $U = \bigcup_{k=0}^\infty B_k$, (2) for every k exists $i_k \in I$ so that $2B_k = B^n(x_k, 2r_k) \subset V_{i_k}$, and (3) $\{k \colon x \in 2B_k\}$ is finite for every $x \in U$.

$\star$**4.** Let $U \subset \mathbb{R}^n$ be an open set and $\mathcal{V} = \{V_i\}_{i\in I}$ an open cover of U. Show that there exists C^∞-functions $\varphi_i \colon U \to [0,1]$, $i \in I$, satisfying the following conditions:
(1) $\mathrm{spt}\varphi_i \subset V_i$ for every $i \in I$, (2) every $x \in U$ has a neihborhood W so that $\{i \in I \colon \varphi_i | W \neq 0\}$ is finite, and (3) $\sum_{i\in I} \varphi_i \equiv 1$.
(*Hint:* Use Problems 2 and 3 to find balls $B_k = B^n(x_k, r_k)$ and functions θ_k so that $B_k = \theta_k^{-1}(0,\infty)$. Consider $\theta = \sum_k \theta_k$ and $\psi_k = \theta_k/\theta$.)

$\star$**5.** Let $f \colon X \to Y$ and $g \colon X \to Y$ homotopic continuous maps between topological spaces.

(i) Show that the map $f_* \colon C_k(X) \to C_k(Y)$ defines a homomorphism $f_* \colon H_k(X) \to H_k(Y)$, $[\sigma] \mapsto [f_*\sigma]$.

(ii) Show that for every 1-cycle $\sigma \in C_1(X)$ there exists a 2-chain $\tau \in C_2(Y)$ so that $f_*\sigma - g_*\sigma = \partial\tau$.

(iii) Conclude[1] that $f_* = g_* \colon H_k(X) \to H_k(Y)$ for $k = 0, 1$.

6. Show that every vector space has a basis. (*Hint:* Given V, consider pairs $(W, e \colon I \to W)$, where $W \subset V$ is a subspace and $(e(i))_{i\in I}$ is a basis of W. Use Zorn's lemma on partial order $<$ defined by
$(W, e \colon I \to W) < (W', e' \colon J \to W')$ iff $W \subset W'$, $I \subset J$, and $e'|I = e$.)

[1] Naturally, this conclusion holds for every k.