Introduction to Differential forms
Spring 2011
Exercise 6 (for Wednesday Mar 2.)

[H] Holopainen, I: "Riemannian geometry", fall 2010.

$\star$**1.** (Section 1.1 in [H]) Using the fact that $\mathbb{S}^n = \{p \in \mathbb{R}^{n+1} \colon |p| = 1\}$ is a manifold, find a smooth structure on $\mathbb{S}^n$. Give an example of a (non-constant) smooth function on $\mathbb{S}^n$.

2. Find a smooth structure on $Q = \partial[0,1]^3 \subset \mathbb{R}^3$. Give an example of a smooth function on Q.

$\star$**3.** (Section 1.21 in [H]) Fill in the details to Theorem 1.22 in [H]. (*Hint:* The topology is given by: a set $V \subset TM$ is open if and only if $\bar{x}(V \cap TU)$ is open in $\mathbb{R}^{2n}$ for every chart $(U, \bar{x})$.)

4. Let N be a smooth submanifold of M. Show that there exists a smooth embedding $I \colon TN \to TM$ so that $\pi_N \circ I = \iota \circ \pi_M$, where $\iota \colon N \to M$ is the inclusion; $\pi_M \colon TM \to M$ and $\pi_N \colon TN \to N$ are projections as usual.

$\star$**5.** (Section 1.9 in [H]) Given a smooth manifold M. One definition of a tangent space of M at x is given as follows. Let $P_p(M)$ be the set of all C^1-paths $\gamma \colon (-\delta, \delta) \to M$ so that $\gamma(0) = p$. Elements of T_pM are equivalence classes $[\gamma]$ of paths $\gamma \in P_p(M)$ so that $\gamma_1 \sim \gamma_2$ if and only if $(\varphi \circ \gamma_1)'(0) = (\varphi \circ \gamma_2)'(0)$ for all charts (U, φ). A candidate for the tangent space of M at p is now $P_p(M)/\sim$.

(i) Show that the formula $\gamma \mapsto \dot{\gamma}$, where $\dot{\gamma} \colon C^\infty(p) \to \mathbb{R}$ is the derivation $\dot{\gamma}(u) = (u \circ \gamma)'(0)$, defines an isomorphism $P_p(M)/\sim \to T_pM$.

(ii) Suppose $M = \mathbb{R}^n$ and let $\hat{T}\mathbb{R}^n$ be the tangent bundle of $\mathbb{R}^n$ as defined in the beginning of the course. Show that the map $\Theta \colon \hat{T}\mathbb{R}^n \to T\mathbb{R}^n$, $(p, v) \mapsto \dot{\gamma}_v$, where $\gamma_v \colon (-1, 1) \to \mathbb{R}^n$ is the path $\gamma_v(t) = p + tv$, defines a bijection that is a linear isomorphisms between tangent spaces $\hat{T}_p\mathbb{R}^n$ and $T_p\mathbb{R}^n$ for every $p \in \mathbb{R}^n$.

6. Using Problems 4 and 5, find the image $\Theta^{-1}T\mathbb{S}^{n-1}$ in $\hat{T}\mathbb{R}^n$ and give it a geometric interpretation.