Introduction to Differential forms
Spring 2011
Exercise 5 (for Wednesday Feb 23.)

$\star$**1.**

(i) Let ω be a compactly supported C^1-smooth $(n-1)$-form in $\mathbb{R}^n$. Show that
$$\int_{\mathbb{R}^n} d\omega = 0.$$

(ii) Let $\alpha \in C^1(\Gamma^k(T\mathbb{R}^n))$ and $\beta \in C^1(\Gamma^\ell(T\mathbb{R}^n))$ be an k- and ℓ-form in $\mathbb{R}^n$, respectively, where $k, \ell \geq 0$ and $k + \ell = n - 1$, so that $\alpha \wedge \beta$ is compactly supported. Show that
$$\int_{\mathbb{R}^n} d\alpha \wedge \beta = -(-1)^k \int_{\mathbb{R}^n} \alpha \wedge d\beta.$$

$\star$**2.** Let (P, ξ), $P = W + p$, be a k-dimensional oriented affine subspace of $\mathbb{R}^n$, $0 < k < n$, and ω a C^1-smooth compactly supported k-form in $\mathbb{R}^n$. Let $v \in \mathbb{R}^n \setminus W$, $W' = \operatorname{span}\{W, v\}$, and $Q = W' + p$. Orient Q with $\xi' = v \wedge \xi$. Show that
$$\int_{P+v} \omega - \int_P \omega = \int_D d\omega,$$
where $D = \{P + tv \colon t \in [0, 1]\} \subset Q$.

$\star$**3.** For every $k \geq 0$ let $\theta_k \colon \bigwedge_k \mathbb{R}^3 \to \operatorname{Alt}^k(\mathbb{R}^n)$ be the standard isomorphism $e_{i_1} \wedge \cdots \wedge e_{i_k} \mapsto \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k}$.

(i) Let $\star \colon \operatorname{Alt}^k(\mathbb{R}^n) \to \operatorname{Alt}^{n-k}(\mathbb{R}^n)$ be the *Hodge star operator* as in Ex. 2 Prob. 4. Define $\star' \colon \bigwedge_k \mathbb{R}^n \to \bigwedge_{n-k} \mathbb{R}^n$ by $\star' = \theta_{n-k}^{-1} \circ \star \circ \theta_k$. Show that the cross product $v \times w$ of vectors in $\mathbb{R}^3$ satisfies $v \times w = \star'(v \wedge w)$ for $v, w \in \mathbb{R}^3$.

(ii) Find a formula for the curl-operator[1] in terms of $\star$, θ and exterior d.

[1] This operator has many names in vector calculus: rot, $\nabla\times$, etc.

(iii) Let X be a C^1-vector field in an open set U of $\mathbb{R}^n$. Show that

$$\operatorname{div}(X) = \star d(X \llcorner dx_1 \wedge \cdots \wedge dx_n),$$

where $X \llcorner dx_1 \wedge \cdots \wedge dx_n$ is the $n-1$-form $(X \llcorner dx_1 \wedge \cdots \wedge dx_n)_p(v_1, \ldots, v_{n-1}) = dx_1 \wedge \cdots \wedge dx_n(X(p), v_1, \ldots, v_{n-1})$.

4.[2] Let $0 < k < n$. Show that there exists a C^∞-smooth $(n-k)$-form ξ so that

$$\int_{\mathbb{R}^k \times \{0\}} \omega = \int_{\mathbb{R}^n} \xi \wedge \omega$$

for all closed compactly supported C^1-smooth k-forms ω in $\mathbb{R}^n$. (*Hint:* Consider forms in $\mathbb{R}^{n-k}$ and the projection $\mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$.)

5. Let $U \subset \mathbb{R}^n$ be an open set and $0 \le k \le n$. The operator $d^* = (-1)^{n(k+1)-1} \star d \star \colon C^1(\Gamma^k(U)) \to C^0(\Gamma^{k-1}(U))$ is called the *co-exterior derivative.*[3]

(i) Show that $d^* \circ d^* = 0$.

(ii) Show that $-\Delta u = d^* du$ for $u \in C^2(U)$, where Δ is the Laplace operator $\Delta u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i \partial x_i}$.

(iii) Define $\Delta = -d^*d - dd^* \colon C^2(\Gamma^k(U)) \to C^0(\Gamma^k(U))$ for $k > 0$. Show that

$$\Delta(u dx_{i_1} \wedge \cdots \wedge dx_{i_k}) = (\Delta u) dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

where $1 \le i_1 < \cdots < i_k \le n$.

6.

(i) Let $Q = [0,1]^3 \subset \mathbb{R}^3$. Show, using the definition, that ∂Q is a manifold.

(ii) Show that

$$T = \{(R + r\cos s)(\cos t, \sin t, 0) + (0, 0, r \sin s) \in \mathbb{R}^3 \colon s, t \in \mathbb{R}\}$$

is a manifold if $R > r > 0$.

[2] Poincaré dual of $\mathbb{R}^k$ in $\mathbb{R}^{n-k}$.
[3] The choice of sign may vary in different texts.