Introduction to Differential forms
Spring 2011
Exercise 2 (for Wednesday Feb 2.)

Problems marked with $\star$ are to be handed at the beginning of the exercise session; problems not marked with $\star$ are discussed in the exercise session.

$\star$**1.** Let V be an n-dimensional vector space.

(i) Let $A = (a_{ij})$ be an $n \times n$-matrix. For every pair $(i, j) \in \{1, \ldots, n\}$ denote by A_{ij} the $(n-1)\times(n-1)$-matrix obtained from A by removing ith row and jth column. Show that the recursive formula

$$\det A = \sum_{k=1}^{n} (-1)^{1+k} a_{1k} \det A_{1k},$$

for the determinant is the same as the formula

$$\det A = \sum_{\sigma \in S_n} \operatorname{sign}(\sigma) a_{1\sigma(1)} \cdots a_{n\sigma(n)},$$

where S_n is the group of permutations of $\{1, \ldots, n\}$.

(ii) Let V be a finite dimensional vectorspace and let $e_1, \ldots, e_n$ be a basis of V and $\varepsilon_1, \ldots, \varepsilon_n$ the corresponding dual basis[1]. Show that

$$\varepsilon_1 \wedge \cdots \wedge \varepsilon_n(v_1, \ldots, v_n) = \det(v_{ij})$$

for $v_1, \ldots, v_n \in V$, where (v_{ij}) is the matrix so that $v_i = \sum_{j=1}^{n} v_{ij} e_j$.

(iii) Prove the Geometric Structure Theorem (see lecture notes).

$\star$**2.** Let $V, (e_1, \ldots, e_n)$, and $(\varepsilon_1, \ldots, \varepsilon_n)$ be as in Problem 1.

(i) Let $\omega \in \mathrm{Alt}^k(V)$. Show that $\omega(v_1, \ldots, v_k) = 0$ if vectors $v_1, \ldots, v_k \in V$ are linearly dependent. Conclude that $\mathrm{Alt}^k(V) = \{0\}$ for $k > \dim V$.

(ii) Let $\omega \in \mathrm{Alt}^k(V)$ and $\tau \in \mathrm{Alt}^\ell(V)$. Show that $\omega \wedge \tau = (-1)^{k\ell} \tau \wedge \omega$.

[1] $\varepsilon_i(e_j) = 0$ for $i \neq j$ and $\varepsilon_i(e_i) = 1$ for all i

3. Let $L\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map so that $\det L = 1$ and $L^*\colon \mathrm{Alt}^2(\mathbb{R}^n) \to \mathrm{Alt}^2(\mathbb{R}^n)$ is the identity. What can we say about L when (i) $n = 2$, (ii) $n = 3$, (iii) $n = 4$?

$\star$**4.** Let V be a finite dimensional vector space with an inner product $\langle \cdot, \cdot \rangle$. Let $(e_1, \ldots, e_n)$ be an orthonormal basis. Let $\Phi\colon V \to V^*$ be the isomorphism satisfying $\Phi(e_i) = \varepsilon_i$, where $\varepsilon_i \in V^*$ is the map defined by $\varepsilon_i(v) = \langle v, e_i \rangle$ ($v \in V$). Denote also $\varepsilon_I = \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_k}$ for all $I = (i_1, \ldots, i_k)$, where $1 \le i_1 < \cdots < i_k \le n$.

(i) For every $k \ge 0$ we define an inner product $\langle \cdot, \cdot \rangle$ on $\mathrm{Alt}^k(V)$ by $\langle \varepsilon_I, \varepsilon_J \rangle = 0$ for $I \ne J$ and $\langle \varepsilon_I, \varepsilon_I \rangle = 1$ for every $I = (i_1, \ldots, i_k)$. Show that

$$\langle \Phi(v_1) \wedge \cdots \wedge \Phi(v_k), \Phi(w_1) \wedge \cdots \wedge \Phi(w_k) \rangle = \det\left(\langle v_i, w_j \rangle \right).$$

for all $v_1, \ldots, v_k, w_1, \ldots, w_k \in V$.

(ii) Show that, for $0 \le k \le n$, there exists a (unique) linear map[2] $\star\colon \mathrm{Alt}^k(V) \to \mathrm{Alt}^{n-k}(V)$ so that

$$\omega \wedge \star\eta = \langle \omega, \eta \rangle \varepsilon_1 \wedge \cdots \wedge \varepsilon_n$$

for $\omega, \eta \in \mathrm{Alt}^k(V)$. (*Hint:* Find first $\star\varepsilon_I$ for all $I = (i_1, \ldots, i_k)$.)

5. Let V and $\star\colon \mathrm{Alt}^k(V) \to \mathrm{Alt}^{n-k}(V)$ be as in Problem 4.

(i) Show that $\star \circ \star = (-1)^{k(n-k)}\mathrm{id}\colon \mathrm{Alt}^k(V) \to \mathrm{Alt}^k(V)$.

(ii) Suppose $A\colon \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal linear map with determinant 1. Show that $A^* \circ \star = \star \circ A^*\colon \mathrm{Alt}^k(\mathbb{R}^n) \to \mathrm{Alt}^k(\mathbb{R}^n)$ for $0 \le k \le n$.

6.

(0) Recall from Topology II the definition of the fundamental group.

(i) Let B be a Euclidean ball $B^n(x, r)$ in $\mathbb{R}^n$ and suppose that $\gamma\colon [0,1] \to B$ is a path. Show that γ is homotopic in B to a path $\hat{\gamma}\colon [0,1] \to B$, $t \mapsto (1-t)\gamma(0) + t\gamma(1)$, relative[3] to $\{0, 1\}$.

(ii) Show that $\pi_1(\mathbb{R}^3 \setminus \{0\}, x_0)$ is trivial for every $x_0 \in \mathbb{R}^3 \setminus \{0\}$. (*Note:* $\mathbb{R}^3 \setminus \{0\}$ is not contractible!)

[2] the "Hodge star"-operator

[3] A homotopy $F\colon Y \times [0,1] \to Z$ is relative to $A \subset Y$ if $F(x,t) = F(x,0)$ for all $x \in A$ and all $t \in [0,1]$.