Introduction to Differential forms
Spring 2011
Exercise 1 (due Wednesday Jan 26.)

Solutions to problems marked with $\star$ are to be handed in at the beginning of the exercise session; problems not marked with $\star$ are discussed in the exercise session.

$\star$**1.** Let $\Omega \subset \mathbb{R}^n$ be an open set and let X and Y be continuous vector fields in Ω. Show that there exists a C^1-function f on Ω so that $X = Y + \nabla f$ if and only if

$$\int_0^1 \langle X(\gamma(t)), \dot{\gamma}(t) \rangle \, \mathrm{d}t = \int_0^1 \langle Y(\gamma(t)), \dot{\gamma}(t) \rangle \, \mathrm{d}t$$

for every C^1-loop[1] $\gamma \colon [0,1] \to \Omega$, $\gamma(0) = \gamma(1)$.

2. Find a C^1-smooth 1-form $\omega \colon \mathbb{R}^2 \setminus \{0\} \to T^*\mathbb{R}^2$ so that $I \colon (0,\infty) \to \mathbb{R}$,

$$r \mapsto \int_{\gamma_r} \omega,$$

is a constant function not equal to zero, where $\gamma_r \colon [0,1] \to \mathbb{R}^2 \setminus \{0\}$ is the path $\gamma_r(t) = (r\cos(2\pi t), r\sin(2\pi t))$.

$\star$**3.** Given $x_0 \in \mathbb{R}^n$ define $P(x_0)$ be the set of C^1-paths $\gamma \colon (-1,1) \to \mathbb{R}^n$ so that $\gamma(0) = x_0$ and define paths $\alpha + \beta \in P(x_0)$ and $a\alpha \in P(x_0)$ by $(\alpha+\beta)(t) = \alpha(t) + \beta(t) - x_0$ and $(a\alpha)(t) = a(\alpha(t) - x_0) + x_0$ for $t \in (-1,1)$. Define an equivalence relation $\sim$ on $P(x_0)$ by $\alpha \sim \beta$ iff $\alpha'(0) = \beta'(0)$ and set $\tilde{T}_{x_0}\mathbb{R}^n = P(x_0)/\sim$. Set also $\tilde{T}E = \bigcup_{x\in E} \tilde{T}_x\mathbb{R}^n$ for any set $E \subset \mathbb{R}^n$.

(i) Show that $\tilde{T}_{x_0}\mathbb{R}^n$ is a vector space with addition $[\alpha] + [\beta] = [\alpha+\beta]$ and scalar multiplication $a[\alpha] = [a\alpha]$.

(ii) Let $\varphi \colon \Omega \to \mathbb{R}^n$ be a C^1-map, where $\Omega \subset \mathbb{R}^m$ is an open set, and let $x_0 \in \Omega$. Show that the map $(\varphi_*)_{x_0} \colon \tilde{T}_{x_0}\Omega \to \tilde{T}_{\varphi(x_0)}\mathbb{R}^n$, $[\gamma] \to [\varphi \circ \gamma]$, is well-defined.

(iii) Show that, for every $x_0 \in \mathbb{R}^n$, there exists an isomorphism $\Phi_{x_0} \colon T_{x_0}\mathbb{R}^n \to \tilde{T}_{x_0}\mathbb{R}^n$ so that $\Phi_{x_1}(D\tau(v)) = \tau_*(\Phi_{x_0}(v))$ whenever $\tau \colon \mathbb{R}^n \to \mathbb{R}^n$ is a translation $x \mapsto x + (x_1 - x_0)$.

[1] A loop is a path with coinciding start and end point.

(iv) Suppose $f\colon \Omega \to \mathbb{R}$ is a C^1-function. Show that $(df)_{x_0}(v) = (f \circ \gamma)'(0)$, where $\Phi(v) = [\gamma] \in \tilde{T}_{x_0}\mathbb{R}^n$ and $x_0 \in \Omega$.

(v) Let V be an affine subspace of $\mathbb{R}^n$ and $x_0 \in V$. Define $\tilde{T}_{x_0}V = \{[\gamma] \in T_{x_0}\mathbb{R}^n \colon \gamma(-1,1) \subset V\}$. Show that $\tilde{T}_{x_0}V$ is well-defined and that $\Phi(T_{x_0}V) = \tilde{T}_{x_0}V$.

4. Let $A\colon \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Show that $m_n(A(E)) = (\det A)m_n(E)$ for all measurable sets $E \subset \mathbb{R}^n$. *Hint:* Elementary matrices and, for example, Rudin: Real and complex analysis.

$\star$**5.** Let V be an n-dimensional vector space and $(e_1, \ldots, e_n)$ a basis of V.

(i) Show that the dual space $V^* = \{f\colon V \to \mathbb{R} \colon f \text{ linear}\}$ has a basis $\{\varepsilon_1, \ldots, \varepsilon_n\}$ so that $f = \sum_{i=1}^n a_i \varepsilon_i$, where $a_i = f(e_i)$, for every $f \in V^*$.

(ii) Let W be an m-dimensional vector space with basis $(e'_1, \ldots, e'_m)$. Find a basis for the space $L(V, W)$ of all linear maps $V \to W$.

6. A topological space X is contractible if there exists a homotopy[2] $F\colon X \times [0,1] \to X$ from id_X to a constant map. Show that $\mathbb{R}^3 \setminus R$, where $R = \{(x,0,0) \colon x \geq 0\} = [0,\infty) \times \{(0,0)\}$, is contractible.

[2]A map $F\colon X \times [0,1] \to Y$ is a homotopy from $f_0\colon X \to Y$ to $f_1\colon X \to Y$ if F is continuous and $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in X$.